Stock market crashes: Some quantitative results based on extreme value theory

Dr François Longin
Director of Research and Innovation, DRI CFF, 75419, Paris Cedex, France. Tel. +33 (0)1 4070 3276; Fax. +33 (0)1 4070 3031; E-mail: flongin@ccf.fr

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François Longin is Director of the Department of Research and Innovation at HSBC CCF and Professor of Finance at ESSEC Graduate Business School.

Abstract
'Stock market crash': a magic expression, which will definitely attract the attention of every financial investor. This paper is the follow-up of a paper published in Derivatives Use, Trading & Regulation *whose objective was to present extreme value theory. This paper now shows how this statistical theory can be used to obtain some quantitative results about such extreme price movements. More precisely, it estimates the probability of an extreme price movement and its waiting time period. It also focuses on extreme price movements associated with stock market crashes. The opinions expressed in this paper are those of the author and do not necessarily reflect the official views of the bank.*

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INTRODUCTION

'Stock market crash': a magic expression, which will definitely attract the attention of every financial investor. Although all market participants would certainly care about such extraordinary events, no research work has ever attempted to give a rigorous quantification of its meaning. In this paper we use extreme value theory to provide some quantitative results on extreme price movements. Using the asymptotic distribution of extreme returns, we compute the probability of a stock market crash and its waiting time period.

This paper is organised as follows: the first part shows how extreme value theory can be used to get information about extreme price movements. Two statistical tools are used: the probability of exceedance of an extreme price movement of a given level and its waiting time period defined as the average time needed to observe an extreme price movement higher than or equal to that level. The second part focuses on stock market crashes. Beyond the statistical properties, the paper also considers qualitative aspects. The conclusion discusses the usefulness of the results in the context of fund management.

EXTREME PRICE MOVEMENTS

In this section, we first recall the definition of extreme price movements in the context of extreme value theory. We then expose the main result given by this statistical theory and the statistical tools used to quantify extreme price movements. Finally, we present some empirical results about the extreme price movements observed in the US equity market.

Definition of extreme price movements

Price movements are measured by the logarithmic returns on the assets or index on a regular basis (a day or a week, for example). The basic return observed on the time-interval $[t-1, t]$ is denoted by R_t. Let $R_1, R_2, \ldots R_T$ be the returns observed over T intervals $[0, 1], [1, 2], [2, 3], \ldots, [T-2, T-1], [T-1, T]$. Extreme price movements can be defined as the maximum and the minimum of the random variables $R_1, R_2, \ldots R_T$. This paper focuses on the minimum denoted by Z_T observed over T trading intervals: $Z_T = \text{Min}(R_1, R_2, \ldots, R_T)$.

Results from extreme value theory

Assuming that returns R_t are independent and drawn from the same distribution F_R, the exact distribution of the minimal return, denoted by F_{Z_T}, is given by

$$F_{Z_T}(z) = 1 - (1 - F_R(z))^T. \tag{1}$$

As noted by Longin,[1] in practice, the distribution of returns is not precisely known and, therefore, if this distribution is not known, neither is the exact distribution of minimal returns. From equation (1), it can also be concluded that the limiting distribution of Z_T obtained by letting T tend to infinity is degenerate: it is null for z less than the lower bound l, and equal to one for z greater than l.

To find a limiting distribution of interest (that is to say a non-degenerate distribution), the minimum Z_T is reduced with a scale parameter α_T (assumed to be positive) and a location parameter β_T such that the distribution of the standardised minimum $(Z_T - \beta_T)/\alpha_T$ is non-degenerate. The so-called extreme value theorem specifies the form of the limiting distribution as the length of the time-period over which the minimum is selected (the variable T) tends to infinity. As shown by Gnedenko,[2] the limiting distribution of the minimal return, denoted by F_Z, is given by:

$$F_z(z) = 1 - \exp(-(1 + \tau \cdot z)^{1/\tau}) \tag{2}$$

for $z < -1/\tau$ if $\tau < 0$ and for $z > -1/\tau$ if $\tau > 0$. The parameter τ, called the tail index, models the distribution tail. According to the tail index value, three types of extreme value distribution are distinguished:[3,4] the Fréchet distribution ($\tau < 0$) obtained for fat-tailed distributions, the Gumbel distribution ($\tau = 0$) obtained for thin-tailed distributions and the Weibull distribution ($\tau > 0$) obtained for bounded distributions. Longin[5] showed that extreme returns in the US equity market seems to obey to a Fréchet distribution.

In practice, as we are interested in the distribution of minimal returns (not standardised extreme returns), the asymptotic distribution denoted by $F_{Z_T}^{asymp}$ depends on the three parameters: the scale parameter α_T, the location parameter β_T and the tail index τ.

Statistical tools

In order to quantify extreme price movements, we use two statistical tools: the probability of exceedance of an extreme price movement of a given level and its associated waiting time period.

For an extreme price movement of a given level z, we compute the probability of exceedance of that level. Denoted by $p^{ext}(z)$, is given by:

$$p^{ext}(z) = 1 - F_{Z_T}^{asymp}(z) = \exp\left[-\left(1 + \tau.\left(\frac{z - \beta_T}{\alpha_T}\right)\right)^{\frac{1}{\tau}}\right] \quad (3)$$

The probability $p^{ext}(z)$ represents the probability to observe a minimal return lower or equal to the level z. Inversely, for a given probability p^{ext}, we can compute the associated level. Denoted by $z(p^{ext})$, it is given by:

$$z(p^{ext}) = -\beta_T + \frac{\alpha_T}{\tau}.[1 - (-\ln(p^{ext}))^{\tau}] \quad (4)$$

The second tool that we use to quantify extreme price movements is the waiting time period. For an extreme price movement of size z, it is defined as the average time to observe a minimal return lower than or equal to that size. Denoted by $T(z)$, it is given by:

$$T(z) = \frac{1}{p^{ext}(z)}. \quad (5)$$

Empirical results

Empirical results are given in Tables 1 to 3.

In Table 1, we give the level of an extreme price movement in the US equity market for a given probability of exceedance or waiting time period. For example, there is a probability equal to 0.50, that over one year we observe a minimal daily return lower than or equal to −2.48 per cent. In other words, we have to wait on average two years to observe a minimal daily return lower than or equal to −2.48 per cent. As the probability decreases, the waiting time period increases and the size of the extreme price movements increases in absolute value. For example, there is a probability equal to 0.05, that over one year we observe a minimal daily return lower than or equal to

Table 1: Level of an extreme price movement in the US equity market for a given probability of exceedance or waiting time period

Probability of exceedance	*Waiting time period*	*Extreme price movement*
0.50	2	−2.48%
0.20	5	−3.99%
0.10	10	−5.65%
0.05	20	−8.02%
0.02	50	−12.87%
0.01	100	−18.51%

Note: this table gives the level of an extreme price movement in the US equity market for a given probability of exceedance or waiting time period. An extreme price movement is defined as the lowest daily return observed over one year. For a given probability of exceedance or waiting time period, the level of the extreme price movement is computed with the extreme value distribution. The database used in the estimation of this distribution consists of daily returns on the S&P 500 index over the period July 1962–December 1999 (9,494 observations). Details of the estimation can be found in Longin.[2]

−8.02 per cent. In other words, we have to wait on average 20 years to observe a minimal daily return lower than or equal to −8.02 per cent.

Table 2 gives the probability of exceedance and waiting time period for a given level of an extreme price movement in the US equity market. For example, a minimal daily return over one year of −5 per cent will be exceeded with a probability equal to 0.127. In other words, we have to wait on average 7.84 years to observe a minimal daily return lower than or equal to −5 per cent. As the level of the minimal return increases in absolute value, the probability of exceedance and the waiting time period increase. For example, a minimal daily return over one year of −10 per cent will be exceeded with a probability equal to 0.032. In other words, we have to wait on average 30.72 years to observe a minimal daily return lower than or equal to −10 per cent.

Table 3 gives the probability of exceedance and waiting time period for the 10th largest extreme price movement in the US equity market. For example, the minimal daily return observed over the year

Table 2: Probability of exceedance and waiting time period for a given level of extreme price movement in the US equity market

Extreme price movement	*Probability of exceedance*	*Waiting time period*
0%	1.000	1.00
−1%	1.000	1.00
−2%	0.711	1.41
−3%	0.351	2.85
−4%	0.199	5.02
−5%	0.127	7.84
−10%	0.032	30.72
−15%	0.014	67.05
−20%	0.009	115.84
−25%	0.006	176.47

Note: this table gives the probability of exceedance and the waiting time period for a given level of extreme price movement in the US equity market. An extreme price movement is defined as the lowest daily return observed over one year. The probability of exceedance represents the probability to observe a minimal return over a year lower than the level. The waiting time period represents the average time period expressed in years needed to observe a minimal return lower of equal to the given level. Both variables are computed with the extreme value distribution. The database used in the estimation of this distribution consists of daily returns on the S&P 500 index over the period July 1962–December 1999 (9,494 observations). Details of the estimation can be found in Longin.[5]

1987 occurred on October 19 and was equal to −18.35 per cent. According to our estimation, such a minimal return should be exceeded with a probability equal to 0.010. In other words, we should have to wait on average 98.40 years to observe a minimal daily return lower than or equal to −18.35 per cent.

EXTREME PRICE MOVEMENTS AND STOCK MARKET CRASHES

A crash certainly corresponds to a minimal return over a given period, but the reverse is not true: a minimal return is not necessarily a crash. For instance, during booming periods, yearly minimal returns are rather small in absolute value: over the

Table 3: Probability of exceedance and waiting time period for the 10th largest extreme price movements observed in the US equity market

Order	*Extreme price movement*	*Date*	*Probability of exceedance*	*Waiting time period*
1	−18.35%	1987/10/19	0.010	98.40
2	−7.11%	1997/10/24	0.063	15.76
3	−7.04%	1998/08/28	0.065	15.46
4	−6.00%	1988/01/08	0.087	11.27
5	−5.72%	1989/10/13	0.098	10.25
6	−4.48%	1986/09/11	0.159	6.29
7	−3.79%	1982/10/25	0.222	4.51
8	−3.55%	1974/11/18	0.254	3.96
9	−3.30%	1991/11/15	0.291	3.43
10	−3.29%	1979/10/09	0.293	3.41

Note: this table gives the probability of exceedance and the waiting time period for the 10th largest extreme price movements observed in the US equity market. An extreme price movement is defined as the lowest daily return observed over one year. The probability of exceedance represents the probability to observe a minimal return over a year lower than the level. The waiting time period represents the average time period expressed in years needed to observe a minimal return lower of equal to the given level. Both variables are computed with the extreme value distribution. The database used in the estimation of this distribution consists of daily returns on the S&P 500 index over the period July 1962–December 1999 (9,494 observations). Details of the estimation can be found in Longin.[5]

year 1964, the largest decline of the US stock market was only −1.38 per cent. This observation is surely not a crash. This raises a natural question: how do we define a crash? In Longin[6] we discuss the definition of a crash and propose two classifications of minimal returns between crashes and non-crashes observations. We then test if heterogeneity in the distribution of the minimal returns could explain the classification between crashes and non-crashes. In other words, we wonder if crashes and non-crashes are drawn from the same unconditional distribution of extremes.

Classification of minimal returns: crashes and non-crashes

Quantitative classification: in the quantitative classification, an observation of minimal return is a crash if the minimal return falls below a given level. To classify the crashes the level can be fixed arbitrarily or determined using a statistical measure such as a multiple of the standard deviations of daily returns.

This classification insists on the quantitative aspect of the phenomenon. A crash corresponds surely to a sharp, brief decline of the market, but the reverse appears to be false. In 1934 for example, the largest decline was −8.15 per cent. This value is the sixth largest drop in the US stock market over more than one century. However, this observation was not recognised as a crash by market participants. Apart from the quantitative characteristics, the crashes also present other aspects (psychological aspects like panic effects or micro-structure aspects like the lack of liquidity, for example), which do not appear in the data. Stock market crashes are 'hard to define but recognizable when encountered'. This leads to a qualitative classification based on market participants' opinion.

Qualitative classification: in the qualitative classification, an observation of minimal return is a crash if it is recognised by market participants (investors, brokers, regulators, etc.) as a crash. We look at the comments reported in the *New York Times* (the only daily newspaper covering the entire period 1885–1990 of the database used in Longin)[6] on the day following the drop in the stock market. If the words *crash* or *to crash* appear in the newspaper, the observation of the minimal return is asterisked as a crash. According to this procedure, we get 14 observations of crashes over 115 observations of minimal returns. The set of crashes can be extended if we take into account synonyms for the word crash. In a less restrictive classification, an observation is called a crash if one of the words crash, to crash, disaster, collapse, to collapse, to tumble and to plunge appear in the articles of the *Times*. The second qualitative classification contains 33 observations of crashes (over 115 minimal returns for the period 1885–1990).

The quantitative and qualitative classifications for the minimal returns are quite different. Although the two sets of crashes approximately contain almost the same number of observations (33 and 31), the overlap is not perfect: there are only 16 observations of crashes common to the two classifications.

Our purpose is to test if the classification of minimal returns between crashes and non-crashes can be explained by heterogeneity in the distribution of the extremes. We wonder if both types of minimal returns are drawn from the same unconditional distribution of extremes or not, or, stated differently, if there is a source of heterogeneity due to crashes.

Methodology

To check if there is heterogeneity due to the crashes in the distribution of extremes, we separate the whole sample of extremes into two subsamples according to our classifications of extremes. For example, the sample of the 115 minimal returns is divided into a subsample containing 31 observations of crashes and a subsample containing 84 observations of non-crashes according to the quantitative classification. Second, we estimate the parameters of the extreme value distribution from the two subsamples and get two sets of the parameters estimates of the unconditional distribution of the minimal returns: one from the sample of crashes noted τ^C, α_n^C and β_n^C and another from the sample of non-crashes noted τ^{NC}, α_n^{NC} and β_n^{NC}. Third, we compare these two sets of estimates. Our null hypothesis corresponds to the homogeneity of the distribution of the minima and can be stated as follows: $\tau_n^C = \tau_n^{NC}$, $\alpha_n^C = \alpha_n^{NC}$ and $\beta_n^C = \beta_n^{NC}$. If the observations of both subsamples are drawn from the same distribution, the null hypothesis should not be rejected.

Empirical results

For the two classifications the results do not lead to a rejection of the null hypothesis of the homogeneity of the distribution of the extremes. There are no significant differences between the parameters estimated from the subsample of the crashes and from the subsample of the other minima. In sum, the results show that the distribution of the extremes is homogenous. The crashes and non-crashes are likely drawn from the same unconditional distribution of extremes. From a statistical point of view, no difference between both types of minimal returns is found. No heterogeneity, which could have explained the classifications of minimal returns was found. The conclusion is that crashes are simply bad draws and not special or abnormal statistical events. Such a result justifies the use of the extreme value distribution to provide quantitative results about stock market crashes.

CONCLUSION

This paper shows that extreme value theory can be useful to quantify the risk of a stock market crash. In particular, we are able to compute the probability of such events and their associated waiting time period.

In practice, such information can be used to define a risk policy of fund management (say pension fund or mutual fund with guarantees). For example, stress tests corresponding to market shocks with a high waiting time period (10, 50 or 100 years) could be defined in order to compute the loss of the assets of the fund. According to the results of the application of these stress tests on the asset (and liabilities as well) of the fund, the manager may reduce the exposition of the fund or hedge the fund against adverse events.

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