Portfolio insurance and market crashes

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Abstract Portfolio insurance has traditionally taken two forms: the buying of put options and the dynamic replication of a given risk profile. While the first method often presents a prohibitive cost and lacks flexibility especially in terms of maturity choice, the second method does not always lead to the expected risk/return profile owing to market imperfections such as market illiquidity. This paper shows how new financial derivatives, referred to here as *crash options*, could be used to protect investors' portfolios during periods of extreme volatility against a sharp, large decline in the position value. A detailed empirical study is carried out for the US stock market using a database of daily return covering the period 1885–1999. Some results are also given for the European stock market over the recent period 1992–2001.

Keywords: *crash option; put option; extreme value theory; stock market crash; portfolio insurance*

Introduction

Market crashes: magic words which will definitely attract the attention of every financial investor. Although all market participants would certainly care about such extraordinary events, no research work has ever attempted to give a rigorous quantification of their meaning. In this paper, extreme value theory is used to specify the distribution of extreme returns observed during stock market crashes. Extreme value theory is a statistical theory which allows one to quantify the behaviour of extreme price movements. Empirically, it is shown that these extraordinary events are well described by the Fréchet distribution. A a new portfolio management instrument called *crash option* is then proposed to insure the portfolio value against a market crash.

Risk is one of the most important factors in the management of financial assets. The efficiency of risk management methods such as portfolio insurance is, however, largely undermined during periods of extreme volatility such as in stock market booms and crashes.[1] As stressed by Rubinstein and Leland (1981), the accuracy of portfolio insurance [see Boulier and Sikorav (1992) for a description of methods based on the replication of options with positions in stock and cash] critically depends on the process followed by the market price: the possibility of gap openings, jump movements and unanticipated changes in volatility may undermine the strategy during periods of market stress. Moreover, portfolio insurance may be liable to have a destabilising effect on the market during these periods.[2] In any event, such a dynamic hedging technique has been used less by market participants since the last stock market crash of October 1987.[3]

To improve the performance of portfolio insurance techniques, especially during periods of market stress, this paper proposes new financial derivatives called crash options whose purpose is to protect the value of a long position against a sharp, large decline in market prices.[4] Portfolio managers could use crash options to limit their extreme downside risk. Crash options complete the spectrum of existing options, as they focus on extreme price changes during a short period. In their conception, crash options are relatively close to lookback options,[5] whose payoff depends on the maximal or minimal price reached by the expiration date. Although lookback options deal with the difference between the price at the expiration date and its maximum or minimum reached during their lifetime, crash options deal with the minimal price *changes* computed over a *short* period of time.

The remainder of this paper is organised as follows: a statistical study about extreme returns is first presented to motivate the introduction of new financial derivatives: it shows that the market is not a Gaussian market but a Fréchet market, as defined in Longin (1996a), characterised by large price movements; a detailed study is also done for extreme returns corresponding to stock market crashes in the second; the third section defines crash options; the fourth section presents the hedge portfolio and the pricing formula in the case of a perfect, continuous Gaussian market used as a benchmark and in an extreme value framework considering the appropriate weight of extremes. The fifth section presents a study based on simulations to show how crash option can be used in portfolio management to get insurance against market crashes. Results are given for both the US and European stock markets. The final section concludes and discusses practical issues related to the trading of crash options.

Extreme returns on the US stock market

This section studies the statistical behaviour of extreme returns using extreme value theory. Empirical results about their distribution are then exploited to characterise the US stock market.

Extreme value theory

Changes in the value of the position are measured by the returns on a regular basis. The basic return observed on the time-interval $[t-1, t]$ of length f is denoted by $r_t\,(f)$. Let us call $F_{r(f)}$ the cumulative distribution function of returns. It can take values in the interval (l, u). For example, for a variable distributed as the normal, $l = -\infty$ and $u = +\infty$. Let $r_1(f)$, $r_2(f)$, …, $r_T(f)$ be the returns observed over T intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, …, $[T-2, T-1]$, $[T-1, T]$. The extreme return denoted by $Z_T(f)$ corresponds to the minimum of the T random variables. The process of selection of extremes is illustrated in Figure 1. The distribution of the minimum denoted by $F_{Z_T(f)}$ is given by

$$F_{Z_T(f)}(z) = 1 - [1 - F_{r(f)}(z)]^T. \qquad (1)$$

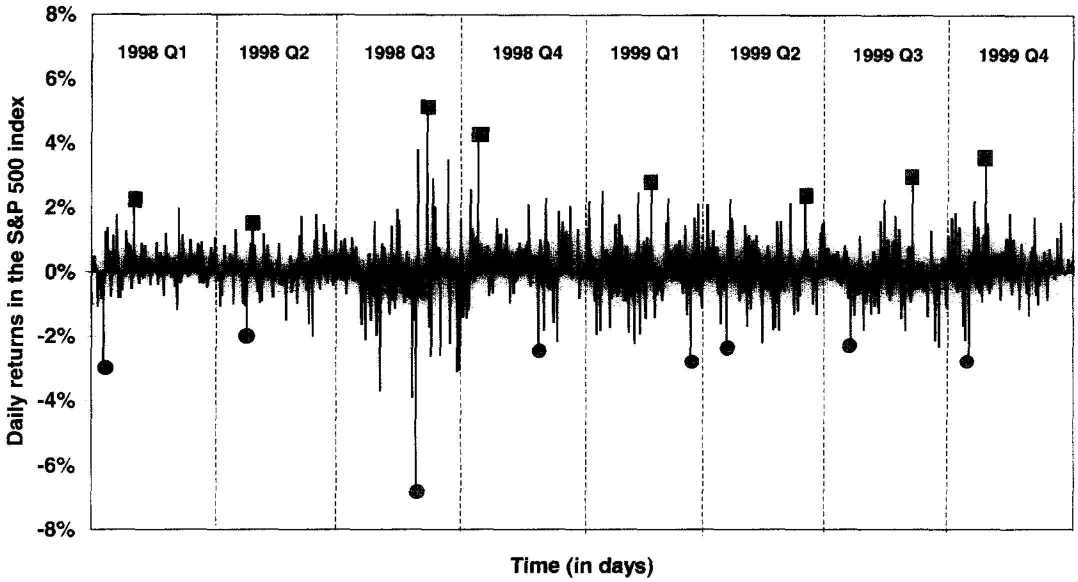

Figure 1 Selection of extreme returns. This figure plots the history of the daily returns in the S&P500 index over the period January 1998–December 1999 containing around 500 observations. Minimal returns (marked by a circle) and maximal returns (marked by a square) are selected over non-overlapping quarters Q1, Q2, Q3 and Q4 every year. The example corresponds to the following parameters' value: $f = 1$ and $T = 63$. From the 500 observations of daily returns, eight observations of extreme returns are obtained. Extreme value theory is mainly concerned with the statistical properties of the extreme observations of the random process.

In practice, the distribution of returns is not precisely known and, therefore, if this distribution is not known, neither is the exact distribution of the extremes. From Equation (1), it can also be concluded that the limiting distribution of $Z_T(f)$ is degenerate. Then, for practical and theoretical purposes, the *asymptotic* behaviour of the extremes is studied. To find a limiting distribution of interest, the maximum variable $Z_T(f)$ is reduced with a (positive) scale parameter $\alpha_T(f)$ and a location parameter $\beta_T(f)$ such that the distribution standardised extremes $[Z_T(f) - \beta_T(f)]/\alpha_T(f)$ is non-degenerate. Gnedenko (1943) proves the so-called *extreme value theorem*, which specifies the form of the limiting distribution F_Z as the length of the period over which extremes are selected (the parameter T tends to infinity. Three possible types of limiting extreme value distributions can be reached:

— The Gumbel distribution:

$$F_Z(z) = 1 - \exp(-e)^{z} \quad \text{for } z \in \Re \qquad (2)$$

— The Fréchet distribution:

$$F_Z(z) = \begin{cases} 1 - \exp(-(-z)^{-\tau}) & \text{for } z < 0 \\ 1 & \text{for } z \geq 0 \end{cases} \quad (\tau > 0) \qquad (3)$$

— The Weibull distribution:

$$F_Z(z) = \begin{cases} 0 & \text{for } z < 0 \\ 1 - \exp(-z^{-\tau}) & \text{for } z \geq 0 \end{cases} \quad (\tau < 0) \qquad (4)$$

The three types of extreme value distribution are represented in Figure 2.

The tail of the distribution $F_{r(f)}$ is either declining exponentially (Gumbel), or by a power (Fréchet) or remains finite (Weibull). For the first and third cases, all

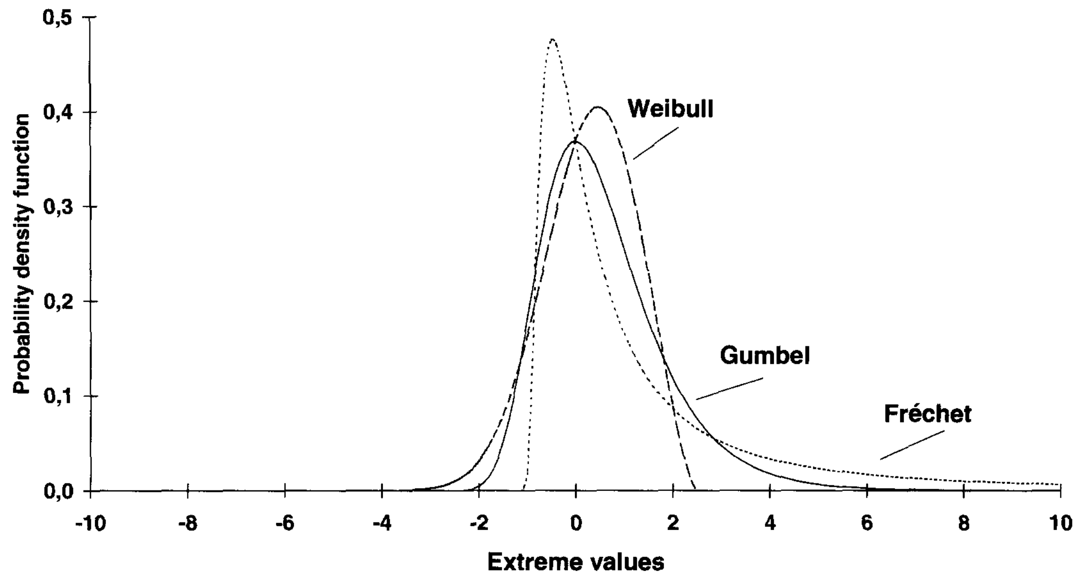

Figure 2 The Fréchet, Gumbel and Weibull extreme value distributions. This figure represents the three types of extreme value distribution that can be distinguished according to the tail index value τ: the Fréchet distribution ($\tau < 0$), the Gumbel distribution ($\tau = 0$) and the Weibull distribution ($\tau > 0$). The distribution for extreme returns is a Fréchet if the distribution of returns is fat-tailed, a Gumbel if the distribution of returns is thin-tailed, and a Weibull distribution if the distribution of returns has no tail (the return and therefore the extreme return are bounded). The tail of the Fréchet distribution ($\tau < 0$) declines slowly at a power rate. The tail of the Gumbel distribution ($\tau = 0$) declines rapidly at an exponential rate. The Weibull distribution ($\tau > 0$) has no tail as there are no observations of returns (nor therefore extreme returns) beyond a certain point. The distributions represented are standardised extreme value distributions ($\alpha_T = 1$, $\beta_T = 0$) with tail index values equal to -0.8 for the Fréchet case, 0 for the Gumbel case and 0.4 for the Weibull case.

moments of the distribution of $r(f)$ are well-defined. For the second case, the shape parameter l reflects the weight of the tail of the distribution of the basic variable $r(f)$: the lower l, the fatter the distribution of $r(f)$. The shape parameter corresponds to the maximal order moment: the moments of order greater than l are infinite and the moments of order less than l are finite: the distribution of $r(f)$ is fat-tailed (Gumbel, 1958: 266). For example, if l is greater than unity, then the mean of the distribution exists; if l is greater than two, then the variance is finite; if l is greater than three, then the skewness is well defined, and so forth. The shape parameter is an intrinsic parameter of the process of returns and does not depend on the number of returns n from which the maximal return is selected.

Jenkinson (1955) proposes a generalised formula (5) which groups the three types distinguished by Gnedenko (1943):

$$F_Z(z) = 1 - \exp\left[-(1 + {}_{\tau}z)^{1/\tau}\right] \quad \begin{cases} \text{for } z < \tau^{-1} \text{ if } \tau < 0 \\ \text{for } z > \tau^{-1} \text{ if } \tau > 0 \end{cases} \qquad (5)$$

The parameter τ, called the tail index, is related to the shape parameter l by $\tau = -1/l$. The tail index determines the type of distribution: $\tau < 0$ corresponds to a Fréchet distribution, $\tau > 0$ to a Weibull distribution, and the intermediate case ($\tau = 0$) corresponds to a Gumbel distribution. The Gumbel distribution can be regarded as a transitional limiting form between the Fréchet and the Weibull distributions as $(1 - \tau z)^{1/\tau}$ is interpreted as

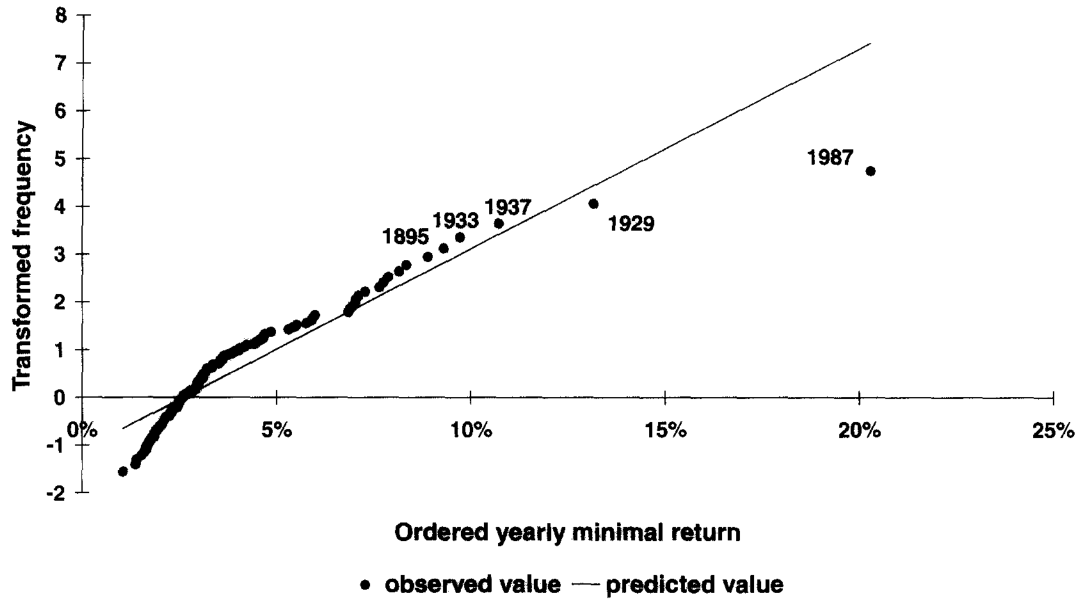

Figure 3 Determination of the type of asymptotic distribution. This figure represents the adjustment of the Gumbel extreme value distribution to observed yearly minimal returns. If minimal returns are drawn from a Gumbel distribution, observed extreme returns should lie on a straight line. Concavity suggests a Fréchet distribution.

e^{-z}. For small values of τ (or large values of l) the Fréchet and Weibull distributions are very close to the Gumbel distribution.

Gnedenko (1943) gives necessary and sufficient conditions for a particular distribution to belong to one of the three types. For example, the normal and log-normal distributions commonly used in finance lead to the Gumbel distribution for the extremes. The Student-t distribution considered by Praetz (1972) obeys the Fréchet distribution with a shape parameter l equal to its degree of freedom ($l \geq 2$). Stable Paretian laws introduced by Mandelbrot (1963) also lead to a Fréchet distribution with a shape parameter l equal to their characteristic exponent ($0 < l < 2$).

The extreme value theorem has been extended to time series: Berman (1964) shows that the same result stands if the variables are correlated (the sum of squared correlation coefficients remaining finite); Leadbetter *et al.* (1983) consider various processes based on the normal distribution: auto-regressive processes with normal disturbances, discrete mixtures of normal distributions as studied in Kon (1984) and mixed diffusion jump processes as advanced by Press (1967) all have thin tails so that they lead to a Gumbel distribution for the extremes; and De Haan *et al.* (1989) show that if $r(f)$ follows the ARCH process introduced by Engle (1982), then the maximum has a Fréchet distribution.

Statistical methods of estimation

Estimated empirically, the asymptotic distribution of extremes contains three parameters only: τ, α_T and β_T. Two parametric techniques are commonly used: the regression method, which provides a graphical method for determining the type of asymptotic distribution, and the maximum-likelihood method, which provides efficient estimates (Figure 3).

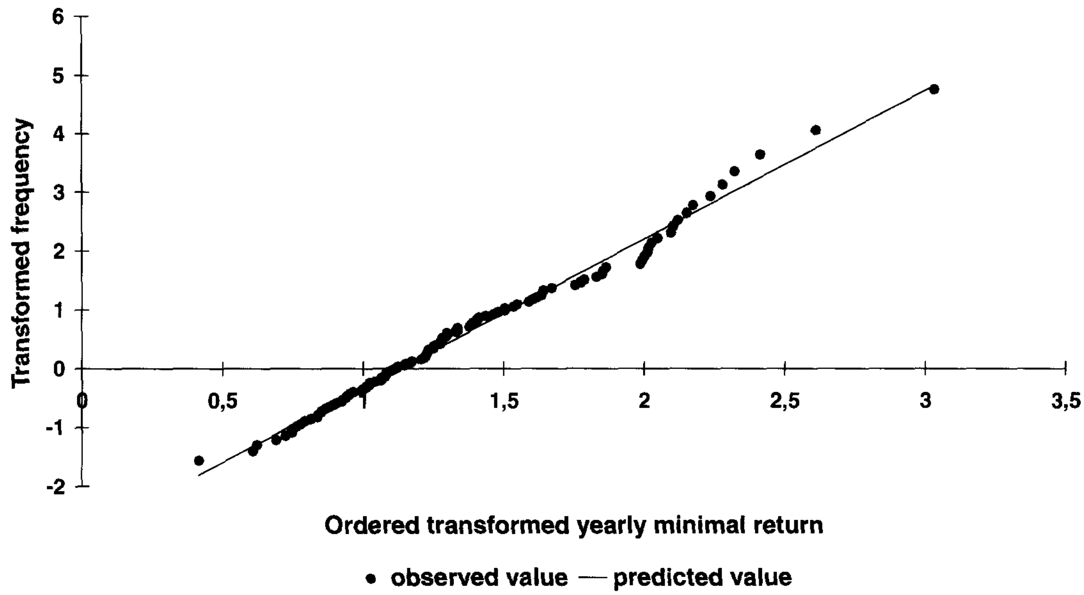

Figure 4 Adjustment of the Fréchet distribution. This figure represents the adjustment of the Fréchet extreme value distribution to observed yearly minimal returns. If minimal returns are drawn from a Fréchet distribution, observed extreme returns should lie on a straight line.

(a) The regression method

The regression method described in Gumbel (1958: 226, 260, 296) is based on order statistics of the extremes Z. The sequence of observed minima $(Z_{n,i})_{i=1,N}$ is arranged in increasing order to get an order statistic $(Z_{T,i})_{i=1,N}$ for which: $Z'_{T,1} \leq Z'_{T,2} \leq \ldots \leq Z'_{T,N}$. For each value of i, the frequency $F_{Z,T}(Z'_{T,i})$ is a random variable lying between zero and one. The distribution of these variables is independent of the variable Z_T. The mean of the ith frequency $E[F_{Z,T}(Z'_{T,i})]$ is equal to $i/(N+1)$. The method compares the ordered extreme observation $F_{Z,T}(Z'_{T,i})$ with its theoretical counterpart $(N+1-i)/(N+1)$. This is done by estimating the reduced Equation (6) obtained by twice taking the logarithm of both quantities:

$$-\ln\left[-\ln\left(\frac{N+1-i}{N+1}\right)\right] = \frac{1}{\tau}\ln\alpha_T - \frac{1}{\tau}\ln_T\left[-\tau\left(Z'_{T,i}-\beta_t+\frac{\alpha_T}{\tau}\right)\right] + \phi_{n,i} \qquad (6)$$

For the intermediate Gumbel case ($\tau = 0$), the following regression is run:

$$-\ln\left[-\ln\left(\frac{N+1-i}{N+1}\right)\right] = \frac{\beta_T - Z'_{T,i}}{\alpha_T} + \phi_{n,i} \qquad (7)$$

Consistent parameter estimates are obtained for both non-linear Equations (6) and (7) by minimising the sum of squared residuals. A graphical test derived by Jenkinson (1955) allows the establishment of a preference for one of the three types of extreme value distribution (see also Gumbel, 1958: 178). The theoretical values $-\ln\{-\ln[(N+1-i)/(N+1)]\}$ are plotted against the observations of ordered extremes $Z'_{T,i}$ on probability paper. The curvature of the resulting graph is related to the type of distribution: for a Gumbel distribution, a straight line should be obtained (Figure 4). The Fréchet distribution leads to a concave curve, while the Weibull

Table 1 Estimation of the asymptotic distribution of minimal daily returns observed over a year

[illegible]	[illegible]	[illegible]	[illegible] τ
		A. [illegible]	
Gumbel distribution	2.378 (0.089)	−2.971 (0.109)	0
Fréchet distribution	1.222 (0.020)	−2.829 (0.016)	−0.395 (0.013)
		B. [illegible]	
Gumbel distribution	1.509 (0.14[illegible])	−2.911 (0.110)	0
Fréchet distribution	1.175 (0.110)	−2.840 (0.124)	−0.337 (0.083)

Note: This table gives parameters' estimates of the distributions of minimal returns based on the regression method (Panel A) and the maximum likelihood method (Panel B). Minimal returns correspond to the lowest daily return reached over a year containing on average 278 trading days over the period 1885–1999. Estimates of the three parameters [$\alpha_T(f)$, $\beta_T(f)$ and τ] are reported for the constrained Gumbel distribution ($\tau = 0$) and for the unconstrained Fréchet distribution.

distribution gives a convex curve. Gumbel (1958: 215) gives the confidence bounds for the graphs.

(b) The maximum-likelihood method

The maximum-likelihood method gives parameter estimators which are unbiased, asymptotically normal and of minimum variance. Parameters' estimates are obtained by solving a set of non-linear equations given by the first-order conditions of the maximisation problem (see Tiago de Oliveira, 1973). Regression estimates are used as initial values in the algorithm. A likelihood ratio test will be computed to discriminate among the three types of asymptotic distributions of extremes.

Empirical results for the US stock market

An extended version of Schwert's (1990) database of returns from 1885 to 1999 is used here. Returns reflect the daily change in the value of an index composed of the most traded stocks on the New York Stock Exchange. Basic returns $r_t(f)$ are computed on a daily basis as percentage logarithmic price change (adjusted for dividends and any change in the capital structure of the firms). Minima $Z_T(f)$ are then defined as the largest daily rise in the stock market and the largest daily fall over a year (containing on average 278 trading days).

Estimates of the scale parameter $\alpha_T(f)$ the location parameter $\beta_T(f)$ and the tail index τ for the distributions of minimal daily returns observed over a year can be found in Table 1, based on the regression method (Panel A) and the maximum-likelihood method (Panel B).

Minimal returns belong to the domain of attraction of the Fréchet distribution as the tail index is significantly negative: −0.337 for minima with an associated *t*-test equal to -4.57. A likelihood ratio test between the Fréchet case and the Gumbel case leads to a firm rejection of the Gumbel distribution (and, *a fortiori*, a rejection of the Weibull distribution). The likelihood value is equal to −240.09 for the Gumbel distribution and −225.08 for the Fréchet distribution The likelihood ratio test (LR) between the two models is asymptotically distributed as a chi-square with one degree of freedom (difference between the number of parameters of each model). The test value is equal to 30.23 with p values less than 10^{-5}.

The goodness of fit of the extreme value distribution can be studied by comparing the empirical frequency of extreme returns with the estimated Fréchet frequency. The empirical

Table 2 Probability of occurrence of minimal returns

	Probability of a minimal daily return lower than r(f)		
Level r(f) (%)	(1) Historical	(2) Fréchet	(3) Gaussian
0	100.00	100.00	100.00
−1	100.00	99.93	100.00
−2	84.34	84.12	99.76
−3	51.30	52.68	29.83
−4	31.30	31.79	0.83
−5	21.33	20.08	0.01
−10	2.60	3.93	0.00
−15	0.86	1.41	0.00
−20	0.86	0.68	0.00
−25	0.00	0.38	0.00

Note: This table gives the probability of a minimal daily return being lower than a given level. A minimal return is defined as the lowest daily return on a portfolio of the most traded stocks on the New York Stock Exchange observed over one trading year. The database contains 115 years covering the period 1885–1999. Three different methods are used to compute the probability: (1) the historical distribution of observed minimal returns; (2) the estimated asymptotic Fréchet extreme value distribution of minimal returns; and (3) the exact extreme value distribution of minimal returns implied by the estimated Gaussian distribution of returns.

frequency of minimal returns is given in Table 2. For example, 51.30 per cent of minimal daily returns observed over a year are under the −3 per cent level and 21.33 per cent under −5 per cent. Such results are close to the one given by the Fréchet distribution which predicts 52.68 per cent of minimal returns under the −3 per cent level and 20.08 per cent under −5 per cent. Such figures differ dramatically from the one predicted by the distribution of extreme returns obtained from a log-normal distribution for daily returns. The results show that the distribution of extremes based on normality fits reality badly; it leads especially to underestimates of the weight of large extreme returns. For example, the probability of a minimal return lower than −5 per cent is 0.01 per cent compared with an empirical frequency of 21.10 per cent.

Characteristics of extreme returns on the US stock market

An economic implication of these results concerns the type of market in which assets are traded by investors. Fama (1963) and McCulloch (1978) discuss two extreme cases: the discontinuous stable Paretian hypothesis and the continuous Gaussian hypothesis. In a stable Paretian market, a large price change over a long time-interval is, most of the time, the result of one or a few very large price changes that took place during smaller subintervals, and the price path contains discontinuities. In a Gaussian market, a large price change is more likely to be the result of many very small price changes, and the price path is continuous. This study of the US market over a long period rejects both hypotheses (the tail index is significantly higher than −0.5 and different from 0) and suggests an intermediate situation (the tail index is between −0.5 and 0). The market under study — *a Fréchet market* — presents more extremes and so more risk for investors than a Gaussian market, but fewer extremes and so less risk than a stable Paretian market. The market price may or may not exhibit discontinuities according to the process governing returns. Such a market characteristic has a direct economic implication for investors following stop-loss, arbitrage or portfolio insurance strategies: in the case of continuity, these strategies may be as reliable as in a Gaussian market, although in practice larger price movements may occur on a short time-interval, and in the case of discontinuity, these strategies may be more efficient than in a stable Paretian market, as large price movements occur less often. In a Fréchet market, investors may have to use specific instruments, such as *crash options* proposed in this paper, to protect their positions during periods of high volatility. The type of market identified by Longin (1996a) is indeed a strong motivation for the introduction of these new financial instruments. Note that a similar conclusion (a Fréchet market) has been reached by Boulier *et al.* (1998) for other financial markets.

Stock market crashes

This section focuses on the crashes in the US stock market. A crash certainly corresponds to a minimal return over a given period, but the reverse is not true: a minimal return is not necessarily a crash. First, the definition of a crash is discussed and two classifications of minimal returns between crashes and non-crashes observations are proposed. Heterogeneity in the distribution of the extremes is then tested as a possible explanation for the classification between the crashes and the non-crashes. In other words, could the crashes and the non-crashes be drawn from the same unconditional distribution of extremes?

Classifications of minima between crashes and non-crashes

Some stock market events have been described as crashes by financial scholars and by professionals. In the previous section, the extreme movements of the stock market defined as minimal returns over a calendar year were studied. If a crash can certainly be characterised as a minimal return over a given period, the reverse is not true: a minimal return is not necessarily a crash. For instance, during booming periods, yearly negative extreme returns are rather small in absolute value: over the year 1985, the largest decline of the US stock market was only −1.47 per cent. This observation is surely not a crash. This raises a natural question: how to define a crash?

Goldsmith (1982) gives the following definition of a financial crisis: 'a sharp, brief, ultracycled deterioration of all or most of a group of financial indicators — short term interest rates, asset (stock, real estate and land) prices, commercial insolvencies and failures of financial institutions'. This study focuses only on one part of the financial crisis: stock market crashes. They are characterised by a sharp, brief decline in stock market prices. This leads us to a quantitative classification based on the size of the extreme returns.

(a) Quantitative classification

In the quantitative classification, an observation of minimal return is a crash if the minimal return falls below a given level. The level can be fixed arbitrarily or determined using a statistical measure. In the empirical study, I used a threshold of 4 standard deviations of daily returns to classify the crashes. The quantitative classification contains 31 observations of crashes (over 115 minimal returns for the period 1885–1999).

This classification insists on the quantitative aspect of the phenomenon. A crash corresponds surely to a sharp, brief decline of the market, but the reverse appears to be false. In 1934, for example, the largest decline was −8.15

per cent. This value is the sixth largest drop in the US stock market over more than one century. This observation was not recognised as a crash by market participants, however. Apart from the quantitative characteristics, the crashes also present other aspects (psychological aspects such as panic effects, or market microstructure aspects such as the lack of liquidity, for example) which do not appear in the data. Aware of the shortcomings of his definition, Goldsmith added the following: crashes are 'hard to define but recognisable when encountered'. This leads us to a qualitative classification based on market participants' opinion.

(b) Qualitative classification

In the qualitative classification, an observation of minimal return is a crash if it is recognised by market participants (investors, brokers, regulators, etc.) as a crash. Consider the comments reported in the *New York Times*[6] on the day following the drop in the stock market. If the words *crash* or *to crash* appear in the newspaper, the observation of the minimal return is asterisked as a crash. According to this procedure, there are 14 observations of crashes over 115 observations of minimal returns. The set of crashes can be extended if we take into account synonyms for the word *crash*. In a less restrictive classification, an observation is called a crash if one of the words *crash*, *to crash*, *disaster*, *collapse*, *to collapse*, *to tumble* and *to plunge* appear in the articles of the *Times*. The second qualitative classification contains 33 observations of crashes (over 115 minimal returns for the period 1885–1999).

The quantitative and qualitative classifications for the minimal returns are quite different. Although the two sets of crashes contain almost the same number of observations (33 and 31), the overlap is not perfect: there are only 16 observations of crashes common to the two classifications.

Test of homogeneity in the distribution of extremes

The purpose of this study is to test whether the classification of minima between crashes and non-crashes can be explained by heterogeneity in the distribution of the extremes. Could both types of minima be drawn from the same unconditional distribution of extremes or not, or, stated differently, if there is a source of heterogeneity due to crashes?

To check whether there is heterogeneity due to the crashes in the distribution of extremes, a regression technique for censored data is used [suggested by Kinnison (1985)]. The regression method can be used for any kind of censored data or any pattern of missing data if the total sample size is known and if the known values can be assigned ranks within the total sample. The method can be described in three steps. First, separate the whole sample of extremes into two subsamples according to our classifications of extremes. For example, the sample of the 115 minimal returns is divided into a subsample containing 31 observations of crashes and a subsample containing 84 observations of non-crashes according to the quantitative classification. Secondly, estimate Equation (7) twice: for the values of i corresponding to the first sample only and for the values of i corresponding to the second sample only. Two sets of the parameter estimates of the unconditional distribution of the minimal returns are obtained: one from the sample of crashes noted τ^{C}, α_T^{C} and β_n^{C} and another from the sample of non-crashes noted τ^{NC}or α^{NC}, and β_T^{NC}. Thirdly, compare these two sets of estimates. Our null hypothesis

Table 3 Estimation of the asymptotic distribution of minimal daily returns corresponding to crashes and non-crashes

[illegible]	[illegible]	[illegible]	[illegible]
A. [illegible]			
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]
B. [illegible]			
[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]

Note: This table gives parameters' estimates of the distributions of minimal returns corresponding to crashes and to non-crashes according to the quantitative classification (Panel A) and the qualitative classification (Panel B).

corresponds to the homogeneity of the distribution of the minima and can be stated as follows: $\tau_n^{C} = \tau_n^{NC}$, $\alpha_T^{C} = \alpha_T^{NC}$ and $\beta_T^{C} = \beta_T^{NC}$. In this paper, a test for each parameter is carried out separately. The statistic of the test is described in Kendall and Stuart (1967). If the observations of both subsamples are drawn from the same distribution, the null hypothesis should not be rejected. A usual issue with this type of test is the power of the test. Is the null hypothesis rejected when the null hypothesis is indeed false? This problem can be particularly important if one of the two subsamples contains very few observations. In this case, the three parameters of the distribution would be badly estimated (that is to say with high standard errors), and finally we would not be able to reject the hypothesis of the coefficients obtained from both subsamples. We would misleadingly conclude that the distribution of the extremes is homogeneous.

A rejection of the hypothesis of homogeneity would suggest that the characteristics of the process describing the daily returns were too far from the assumptions of the extreme value theorem. Series of normalising coefficients (α_T and β_T) would not exist, and the extremes would not be asymptotically drawn from an unconditional distribution.

Empirical results

Our empirical results are presented in Table 3 for the quantitative classification (Panel A) and for the qualitative classification (Panel B).

For the two classifications, the results do not lead to a rejection of the null hypothesis of the homogeneity of the distribution of the extremes. No significant differences are found between the parameters estimated from the sample of the crashes and from the sample of the other minima. For example, for the quantitative classification of the minima, the tail index τ is equal to -0.421 for the crashes and to -0.361 for the other negative extremes. The standard errors for these estimates are respectively equal to 0.101 and 0.034. They are naturally higher than that obtained from the whole sample (0.013) since fewer observations are used for the estimation. The test of the equality of the coefficient τ for the two subsamples leads to a t ratio equal to 0.093 with an associated p value of 0.930. The hypothesis of equality of the two parameters is far from being rejected. All estimates are indeed very close, and the hypothesis of homogeneity could not be rejected even if the parameters were better estimated.

In sum, the results show that the distribution of the extremes is homogeneous. The crashes and non-crashes are likely to be drawn from the same unconditional distribution of extremes. From a statistical point of view, no difference between both types of minima is found. No heterogeneity which could have explained the classifications of minima was found. The conclusion is that crashes are simply bad draws and not special or abnormal statistical events.

Definition of crash options

Options are completely defined when the underlying interest, the payoff function depending on the strike and the expiration date, the type of exercise and the settlement procedure are specified. These attributes are defined below for crash options. The use of such instruments during the crash of October 1987 is studied as an example.

Underlying interest

There are, *a priori*, no conditions to impose on the choice of the underlying interest. While this paper focuses on the equity market, commodities, foreign exchange and bond markets may be considered as well. Crash options could be indifferently associated with individual stocks or any portfolios of these assets combined together. As already noted by Cox and Rubinstein (1985: 446–58), however, financial theory suggests that options written on asset portfolios are potentially of greater social usefulness than conventional options written on single equity securities. Stock and futures indexes[7] such as the S&P100, S&P500, S&P MidCap indexes, the Nasdaq-100 index, the Major Market index, the NYSE index, and the Value Line index in the USA are potential good candidates. As an example used throughout the paper, a portfolio composed of the stocks of the Standard & Poor's 500 index is considered. An investor with a long position will be sensitive to a large decline in the S&P500 index price and will protect his position with a crash option to limit his extreme downside risk.

Payoff function

The aim of crash options is to protect a position against a sharp variation in market prices over a short period of time. As noted by Kindleberger (1978), stock market crashes generally occur over a few days. The period of reference denoted by f could then range from a day to a few weeks. As it is the *change* in the value of the underlying interest which matters, we thus consider percentage price changes or returns.

As a consequence, the strike is also defined as a return. The striking return should be computed in relation to the volatility of the price of the underlying interest. For example, S&P500 index returns present a standard deviation around 1 per cent in daily units. Considering a crash option with a daily frequency to protect a long position in the S&P500 portfolio, striking returns of 0 per cent, −1 per cent, −2 per cent, −3 per cent, −4 per cent and −5 per cent are potential good candidates. A crash option with a 0 per cent striking return annihilates the impact of the largest daily drop on the portfolio value, while a crash option with a −3 per cent striking return limits the impact of the largest daily drop to −3 per cent (if the S&P500 index dropped by more than 3 per cent on a single day by the expiration date). The striking return is denoted by k.

As the strike is expressed as a rate of return, it is necessary to define the notional value associated with crash options contracts. This represents the amount of money initially protected by crash options. The notional value of a standard contract is denoted by NV.

As the type of event related to crash options is essentially rare, the time to expiration of these options should be long enough that the probability of such an event is non-negligible. The time to expiration could range from a few months to a few years. The time to expiration is denoted by T and expressed in the unit of the chosen frequency f.

The payoff for a crash option issued at time 0 with striking return k, notional value NV, and frequency f expiring at date T, is defined by

$$NV \text{ Max } (k - Z_T(f), 0). \quad (8)$$

Loosely speaking, returns $r_1, r_2, r_3, \ldots, r_{T-1}, r_T$ observed over the n basic time-intervals $[0, 1]$, $[1, 2]$, $[2, 3]$, …, $[T-2, T-1]$, $[T-1, T]$ are compared with the striking return k. If there is a time-interval $[t-1, t]$ when the difference $k - r_t$ is positive (that is to say if the price of the underlying interest dropped by more than k per cent during the time-interval $[t-1, t]$), then the owner of the crash option will receive *for sure* at expiration date T a positive amount of money at least equal to $NV(k - r_t)$. The exact amount of money received at expiration date T is equal to $NV(k - r_t^\star)$, where $r_t^\star$ is the lowest return (assumed to be lower than k) occurring during the basic time-interval $[t^\star - 1, t^\star]$. If none of the returns $r_1, r_2, r_3, \ldots, r_{T-1}, r_T$ is lower than the striking return k, then the owner of the crash option receives nothing at the expiration date.

The payoff of a crash option evolves with time according to the path followed by the price of the underlying interest.

Other features

As for other options, the type of exercise of crash options may be either European or American. A third possibility may be to allow the buyer to cash in sequentially the sure value instead of awaiting the expiration date. Unlike European-type options, investors would receive cash during periods of high volatility when it is sometimes badly needed, and, unlike American-type options, investors would not have to give up potential gains that can be obtained until the expiration date.

Purchasers of crash options would simply pay the premium at the beginning of the transaction, and writers would have to deposit cash and securities with their broker or the Exchange as collateral for the writer's obligation to buy or sell the underlying interest. The level of margin requirement should be related to the volatility of the underlying interest and especially be in line with the frequency of extreme price movements.[8]

The trading of crash options may be easier in a discrete-time market, organised for example after the close of the market of the underlying interest, than in a continuous market. The liquidity of the market for crash options may be better after the close and then reliable official closing prices could be used to define the price change or return.

The settlement procedure for crash options can copy the procedure for other options: physical delivery in the case of an individual stock, and cash payment in the case of an index.

An elementary property of crash options

As options with a cliquet, crash options contain a sure value and this sure value increases over the lifetimes of the options. This result comes from a simple mathematical property of the extremes:

these variables are monotone. For example, the minimum of random variables decreases with the length of the period over which it is selected.

Let $CO(t, k, NV, f, T)$ denote the value at time t of a crash option issued at time 0 with striking return k, notional value NV, frequency f and time to expiration T. This value can be written as the sum of two terms

$$CO(t,k,NV,f,T) = e^{-r(T-t)} NV\_\mathrm{Max}[k - Z_t(f), 0] + CO\{t, \mathrm{Min}[k,Z_t(f)], NV, f, T-t\}. \quad (9)$$

The first term is the discounted value of a bond that pays NV $\mathrm{Max}[k - Z_t(f),0]$ at expiration date T (the money the investor is sure at time t to receive at expiration date T from the crash option). The second term is the value of a new crash option issued at time t with notional value NV, frequency f, time to expiration $T-t$ and an 'updated' striking return $\mathrm{Min}[k,Z_T(f)]$.

Equation (9) allows us to determine the optimal time to exercise an American crash option: $(e^{r(T-t)} - 1)$ NV $\mathrm{Max}[k - Z(f),0] \geq CO\{t,\mathrm{Min}[k,Z_t(f)], NV,f,T-t\}$, a condition expressing that the crash option should be exercised when the interests earned on the sure value invested in a risk-free bond are higher than the expected future gains that could be made from the crash option. This condition should be verified after a large negative return, making the sure value high and the price of the new crash option low.

An example

Consider an investor with a portfolio composed of stocks of the S&P500 index (long position) during October 1987. The initial value of his portfolio is \$1,000,000. Let us assume that the investor had the inspired idea of protecting his portfolio with a crash option to limit the impact of a potential stock market crash on his portfolio value. At the beginning of October, he bought a crash option written on the S&P500 index with a striking return of 0 per cent, a notional value of \$1,000,000, a daily frequency and a time to expiration of one month. This option is aimed at removing from the performance of the portfolio the worst daily loss resulting from the largest daily price decline in the S&P500 index during October 1987.

During October 1987, percentage daily returns on the S&P500 portfolio were as follows: 1.71 (1st October), 0.23, 0.00, −2.70, −0.21, −1.38, −0.98, −0.54, 1.66, −2.95, −2.34, −5.16, −20.46 (19th October), 5.33, 9.10, −3.92, −0.01, −8.28, 2.42, 0.04, 4.93, and 2.87 (30th October) as represented in Figure 5. On 6th October, the S&P500 index price dropped by 2.70 per cent; the owner of the crash option is then sure to receive at least \$27,000 at the expiration date. This number is equal to the difference between the striking return (0 per cent) and the minimal return reached on 6th October (−2.70 per cent) times the notional value (\$1,000,000). From 6th October on, the money that he will get for sure from his crash option will increase in value only if the market drops during a single day by more than 2.70 per cent (the updated striking return). A few days later, on 14th October, the price index dropped further, by 2.95 per cent. The sure value of the crash option is now \$29,500 [the difference between the striking return (0 per cent) and the minimal return reached on 14th October (−2.95 per cent)

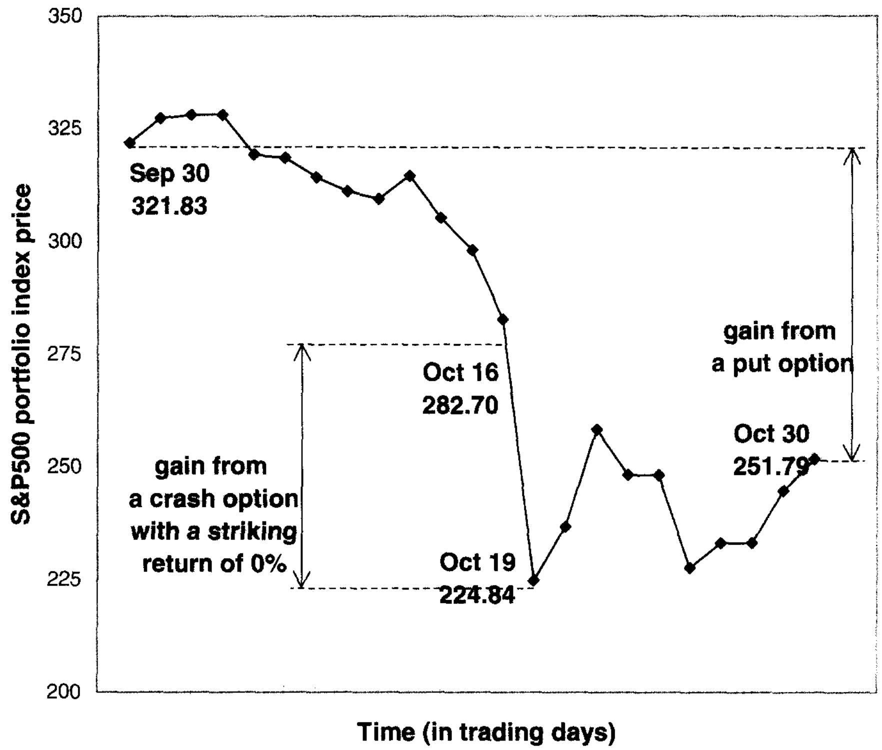

Figure 5 Comparison between a put option, a lookback option on the maximum and a crash option: the case of October 1987. This figure represents the evolution of the S&P500 index in October 1987 and the payoff of a put option and a crash option written on the index.

times the notional value ($1,000,000)]. Larger negative returns occurring during the October crash lead to future increases in the crash option sure value. During October 1987, the minimum variable Z_{22} is equal to the -20.46 per cent reached on 19th October. At the expiration date, the crash option is finally worth $204,600. Although the market dropped by 3.92 per cent and 8.28 per cent (returns lower than the crash option striking return of 0 per cent) on 22nd and 26th October, respectively, this does not increase the sure value, since these returns are greater than the updated striking return of -20.46 per cent, the minimal return reached by these dates. In this example, the use of a crash option removes the impact of the crash (-20.46 per cent) from the monthly portfolio performance. The return on the investment during October 1987 is improved by around 20 per cent by using the crash option: during October 1987 the value of the unprotected portfolio dropped by 21.75 per cent while the value of the same portfolio protected by the crash option dropped by only 1.29 per cent.

Hedge portfolio and pricing formula of European crash options in a perfect market

Hedging and pricing are first considered in the classical case of a perfect, continuous Gaussian market. The hypothesis of normality is then relaxed; a more general pricing formula using the asymptotic extreme value distribution is then proposed to take into account the right frequency of extreme returns.

The classical approach

As stated in Black and Scholes (1973), the following assumptions are made: the short-term interest rate is known and is constant through time, and denoted r; the price of the underlying interest follows a random walk in continuous time with a variance proportional to the square of the asset price. The asset price denoted as S is governed by a geometric Brownian motion given by

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{10}$$

where the drift μ and the variance σ^2 are assumed to be constant over time; the price of the underlying interest takes into account dividends paid on the stock (or the stocks included in the portfolio) and any distributions related to the change in the capital structure of the firm(s); there are no transaction costs of buying or selling stocks and bonds; investors can borrow any fraction of the price of a security to buy or hold it, at the short-term interest rate; and there are no penalties for short selling.

The hedge portfolio for a crash option can be structured as a string of forward start put options (with random strike prices). To cover the interval $[t, t+1]$, the investor buys, at time t, $NV^{-r(T-t)}/S_t$ put options on the underlying interest with striking price $S_t\{1 + \mathrm{Min}[k, Z_t(f)]\}$ and maturity f (ie expiring at time $t+1$), whose value is denoted by $P\{S_t, S_t[1 + \mathrm{Min}(k, Z_t(f))], f\}$. At time 0, a part of the proceeds from the sale of the crash option is used to finance a portfolio used to buy these put options. From the point of time 0, this portfolio can be viewed as a contingent claim, whose value at the expiration date (time t) depends only on the price of the underlying interest. The value of the portfolio denoted by V_{t+1} is the solution to the Cauchy problem with the partial differential equation

$$-\frac{\partial V_{t+1}}{\partial_t} + rV_{t+1} = rS\frac{\partial V_{t+1}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_{t+1}}{\partial S^2} \tag{11}$$

subject to the boundary condition

$$V_{t+1}(t) = NVe^{-r(T-t)} \frac{1}{S_t} P\{S_t, S_t[1 + \mathrm{Min}(k, Z_t(f))], f\}.$$

The exact composition of the portfolio is determined by computing the delta of the position equal to the partial derivative of the portfolio value with respect to the price of the underlying interest, $\partial V_{t+1}/\partial S$. Over the time-interval $[0, t]$, the position is short in the underlying interest as an increase in the price of the underlying interest may decrease the price of put options to buy at time t.[9] At time t, the composition of the portfolio is changed to buy put options. Over the time-interval $[t, t+1]$, the portfolio of put options is then hedged with a short position in the underlying interest and a long position in the risk-free bond, as given in Black and Scholes (1973). At time $t+1$, the proceeds of put options bought at time t are invested in cash until the time to expiration of the crash option.

At any time, the composition of the portfolio replicating the crash option is given by the sum of the deltas of the n contingent portfolios

$$\frac{\partial V_1}{\partial S} + \frac{\partial V_2}{\partial S} + \frac{\partial V_3}{\partial S} + \ldots + \frac{\partial V_{T-1}}{\partial S} + \frac{\partial V_T}{\partial S},$$

and the price of the crash option is equal to the sum of the values of the n contingent portfolios $V_1 + V_2 + V_3 + \ldots + V_{T-1} + V_T$. The price of the crash option can also be computed by using Equation (6) of the final payoff of the crash option

$$CO(0,k,NV,f,T) = e^{-rT} NVE_{Q_{Z_T(f)}} \{\text{Max}\ [k - Z_T(f),0]\}, \quad (13)$$

where $Q_{Z_T(f)}$ is the risk-neutral distribution of the minimal return observed on the n basic time-intervals [0, 1], [1, 2], [2, 3], ..., $[T-2, T-1]$, $[T-1, T]$, each time-interval being of length f. According to Equation (1), this distribution is related to the nth power of the risk-neutral distribution of a return $r(f)$, $Q_{r(f)}$. As the price of the underlying interest follows a Brownian motion, the latter distribution is a log-normal distribution with mean $e^{rf} - 1$ and variance $e^{2rf}(e^{\sigma^2 f} - 1)$.

The hedging strategy and pricing method are now illustrated with an example. As in the example in the previous section, an investor with a long position in the S&P500 index during October 1987 is considered. The initial value of the portfolio is $1,000,000. At the beginning of October, the investor bought a European crash option written on the S&P500 index with a striking return of 0 per cent, a notional value of $1,000,000, a daily frequency and a time to expiration of one month. This crash option annihilates the impact of the largest daily decline in the index price on his portfolio value. The time-evolution of the crash option value and of the hedge portfolio are given in Table 4. Also given are the value and the decomposition of the basic portfolios of the hedge portfolio: the portfolios used to replicate the crash option over the period $[0, t]$, whose combined values are equal to the discounted sure value of the crash option; the value of the portfolio of one-day put options used to replicate the crash option during the period $[t, t+1]$; and the portfolios used to buy put options over the remaining period $[t+1, T]$. In a perfect Gaussian market, the crash option bought in the beginning of October is worth $19,137.39. It is hedged with a short position in stocks of −$49,017.68 and a long position in the risk-free bond of $68,155.06. The crash option can also be decomposed as a portfolio of put options worth $3,983.32 (equivalent to a short position in stocks of −$490,237.69 and a long position in the risk-free bond of $494,221.01) and contingent portfolios used to buy put options in the future worth $15,154.06 (equivalent to a long position in stocks of $441,220.01 and a short position in the risk-free bond of −$426,065.94). At the expiration date, the crash option is finally worth $204,600.00. By comparison, a put option and a lookback option on the maximum issued at the beginning of October to protect a portfolio of $1,000,000 would have been worth at the end of the month, respectively, $217,630.43 and $237,050.62. Note that such options would have been worth almost nothing at maturity if the market had come back around its pre-crash level, while a crash option would not have been influenced by the post-crash price history as it contains a cliquet. Because of the crash of 19th October, 1987, the crash option finishes deeply in the money.[10]

The extreme value approach

As suggested in the second section, the distribution of market price changes may be different from the Gaussian distribution. In particular, looking at extreme price changes, the Gumbel type of extreme value distribution, which is implied by normality, is strongly rejected in favour of the Fréchet type consistent with fat tails. A more general approach using the asymptotic extreme value distribution is then proposed to take into account the right frequency of extreme

Table 4 Price and hedge portfolio of a European crash option during October 1987 (assuming a perfect Gaussian market)

(1) S&P500 index price at time t	(2) Return $r_t(f)$ [Updated striking return Min (k, $Z_t(f)$)] (%)	(3) Crash option dollar value (Stocks; Bonds)	Dollar value and decomposition of the basic portfolios of the hedge portfolio: (4) Time-interval [0, t] (Stocks; Bonds)	(5) Time-interval [t, $t+1$] (Stocks; Bonds)	(6) Time-interval [$t+1$, T] (Stocks; Bonds)
30th Sep. 1987	–	19,137.39	0.00	3,983.32	15,154.06
321.83	[0.00]	(−49,017.68; 68,155.06)	(0.00; 0.00)	(−490,237.69; 494,221.01)	(441,220.01; −426,065.94)
1st Oct. 1987	+1.71	18,932.00	0.00	3,983.95	14,948.05
327.33	[0.00]	(−50,399.75; 69,331.75)	(0.00; 0.00)	(−490,314.40; 494,298.35)	(439,914.65; −424,966.59)
2nd Oct. 1987	+0.23	18,712.30	0.00	3,984.57	14,727.73
328.07	[0.00]	(−52,992.05; 71,704.35)	(0.00; 0.00)	(−490,391.13; 494,375.70)	(437,399.08; −422,671.35)
5th Oct. 1987	+0.00	18,492.73	0.00	3,985.19	14,507.54
328.08	[0.00]	(−55,348.23; 73,840.96)	(0.00; 0.00)	(−490,467.87; 494,453.06)	(435,119.64; −420,612.10)
6th Oct. 1987	−2.70	27,117.31	26,924.06	10.82	182.43
319.22	[−2.70]	(−3,354.01; 30,471.32)	(0.00; 26,924.06)	(−3,453.82; 3,464.64)	(99.81; 82.62)
7th Oct. 1987	−0.21	27,109.97	26,928.27	10.82	170.88
318.54	[−2.70]	(−3,361.62; 30,471.59)	(0.00; 26,928.27)	(−3,454.90; 3,465.72)	(92.74; 78.14)
8th Oct. 1987	−1.38	27,102.74	26,932.49	10.82	159.44
314.16	[−2.70]	(−3,370.27; 30,473.02)	(0.00; 26,932.49)	(−3,454.90; 3,465.72)	(84.63; 74.81)
9th Oct. 1987	−0.98	27,096.80	26,936.70	10.82	149.28
311.07	[−2.70]	(−3,376.38; 30,473.19)	(0.00; 26,936.70)	(−3,455.44; 3,466.27)	(79.06; 70.22)
12th Oct. 1987	−0.54	27,090.02	26,940.92	10.82	138.28
309.39	[−2.70]	(−3,380.63; 30,470.66)	(0.00; 26,940.92)	(−3,455.98; 3,466.81)	(75.35; 62.93)
13th Oct. 1987	+1.66	27,084.50	26,945.13	10.83	128.54
314.52	[−2.70]	(−3,384.56; 30,469.06)	(0.00; 26,945.13)	(−3,456.52; 3,467.35)	(71.96; 56.58)
14th Oct. 1987	−2.95	29,499.57	29,444.66	4.62	50.29
305.23	[−2.95]	(−1,560.49; 31,060.06)	(0.00; 29,444.66)	(−1,574.50; 1,579.12)	(14.02; 36.27)
15th Oct. 1987	−2.34	29,498.62	29,449.27	4.62	44.73
298.08	[−2.95]	(−1,564.66; 31,063.29)	(0.00; 29,449.27)	(−1.574.75; 1.579.37)	(10.09; 34.65)
16th Oct. 1987	−5.16	51,419.48	51,419.48	0.00	0.00
282.70	[−5.16]	(−0.10; 51,419.58)	(0.00; 51,419.48)	(−0.10; 0.10)	(0.00; 0.00)
19th Oct. 1987	−20.46	204,312.07	204,312.07	0.00	0.00
224.84	[−20.46]	(0.00; 204,312.07)	(0.00; 204,312.07)	(0.00; 0.00)0.00	(0.00; 0.00)
20th Oct. 1987	+5.33	204,344.04	204,344.04		0.00
236.83	[−20.46]	(0.00; 204,344.04)	(0.00; 204,344.04)	(0.00; 0.00)	(0.00; 0.00)
21st Oct. 1987	+9.10	204,376.02	204,376.02	0.00	0.00
258.38	[−20.46]	(0.00; 204,376.02)	(0.00; 204,376.02)	(0.00; 0.00)	(0.00; 0.00)

Table 4 (continued)

(1) S&P500 index price at time t	(2) Return $r_t(f)$	(3) Crash option dollar value	Dollar value and decomposition of the basic portfolios of the hedge portfolio		
	[illegible]	(Stocks; Bonds)	(4) Time-interval [0, t] (Stocks; Bonds)	(5) Time-interval [t, t + 1] (Stocks; Bonds)	(6) Time-interval [t + 1, T] (Stocks; Bonds)
22nd Oct. 1987 248.25	[illegible] [illegible]	204,408.00 (0.00; 204,408.00)	204,408.00 (0.00; 204,408.00)	0.00 (0.00; 0.00)	0.00 (0.00; 0.00)
[illegible]	[illegible]	[illegible]	[illegible]	0.00 (0.00; 0.00)	0.00 (0.00; 0.00)
[illegible]	[illegible]	[illegible]	[illegible]	0.00 (0.00; 0.00)	0.00 (0.00; 0.00)
[illegible]	[illegible]	[illegible]	[illegible]	0.00 (0.00; 0.00)	0.00 (0.00; 0.00)
[illegible]	[illegible]	[illegible]	[illegible]	0.00 (0.00; 0.00)	0.00 (0.00; 0.00)
[illegible]	[illegible]	[illegible]	[illegible]	0.00 (0.00; 0.00)	0.00 (0.00; 0.00)
[illegible] 251.79	[illegible]	[illegible] (0.00; 204,600.00)	[illegible] (0.00; 204,600.00)	0.00 (0.00; 0.00)	0.00 (0.00; 0.00)

Note: This table gives the price and the hedge portfolio of a European crash option during October 1987. A crash option written on the S&P500 index with a striking return of 0 per cent, a notional value of $1,000,000, a daily frequency *and a time to expiration of one month is considered*. The crash option is first proposed for trading at the end of the day of 30th September, 1987. The expiration date is 30th October, 1987. The first column recalls the S&P500 index price at the end of day *t*. The second column gives the return $r_t(f)$ observed during time-interval $[t-1, t]$ and the updated *striking return of the crash option* below in brackets. The third column presents the dollar value of the crash option at the end of each day *t* and the decomposition of the hedge portfolio in stocks and bonds below, in parentheses, assuming a perfect market and a geometric Brownian motion for the index price. A Monte Carlo method using 1,000,000 simulations is used *to compute the price and* the delta of the crash option. The short-term rate is assumed to be constant and equal to 4.35 per cent per year. The variance is also assumed to be constant and equal to 17.02 per cent in yearly units. The last three columns provide the dollar value and the decomposition of the basic portfolios of the hedge portfolio as presented *in the third section*. Column (4) gives the total value of the portfolios used to replicate the crash option over the time-interval $[0, t]$, which is equal to the discounted sure value of the crash option. Column (5) gives the value of the portfolio of one-day put options used to replicate the crash option during the time-interval $[t, t+1]$. *Column (6) gives the value of the portfolios used to buy put options over the remaining time-interval* $[t+1, T]$.

returns, which is of particular importance for pricing crash options. In this section, we still assume a perfect, continuous market. Crash options can be perfectly hedged and then be priced as if they existed in a risk-neutral world, as shown by Harrison and Kreps (1981).

Assuming all risks can be diversified, the asymptotic value at time 0 of a crash option with striking return k, notional value NV, frequency f, and time to expiration T, is given by

$$CO(0,k,NV,f,T=) = e^{-rT} NV E_{Q_{Z_T(f)}}^{\text{asymptotic}} \{\text{Max}\,[k - Z_T(f), 0]\}, \quad (14)$$

where $Q_{Z_T(f)}^{\text{asymptotic}}$ is the risk-neutral asymptotic extreme value distribution of the minimal returns observed on the n basic time-intervals [0, 1], [1, 2], [2, 3], ..., $[T-2,\ T-1]$, $[T-1,\ T]$, each time-interval being of length f. This distribution is given by the formula

$$Q_{Z_T(f)}^{\text{asymptotic}}(z) = 1 - \exp\left\{-\left[1 - \tau^*\left(\frac{\beta_T^*(f)}{\alpha_T^*(f)}\right)\right]^{1/\tau^*}\right\}$$

$$\text{for } z < \frac{\alpha_T^*(f)}{\tau} + \beta_T^*(f), \quad (15)$$

where the parameters $\alpha_T^*(f)$, $\beta_T^*(f)$ and τ^* are the risk-adjusted scale and location parameters and the risk-adjusted tail index. In order to get a finite price for the crash option, the distribution of returns has to belong to the domain of attraction of the extreme value distribution with a tail index greater than minus one.

The extreme value approach includes the classical approach as a particular case: the risk-adjusted parameters are given by $\alpha_T^*(f) = \alpha_T(f)$, $\beta_T^*(f) = \beta_T^*(f) - (\mu f - rf)$ and $\tau^* = \tau = 0$.[11] The risk-neutral distribution differs from the historical distribution by the location parameter value only: the location parameter of the risk-neutral asymptotic distribution of extremes, $\beta_T^*(f)$, is simply equal to the location parameter of the historical asymptotic distribution of extremes $\beta_T(f)$ minus the risk premium observed on a short basic time-interval, $\mu f - rf$. The scale parameter and the tail index are unaffected by the change in distribution.

As pricing Equation (13) is asymptotic, the time to expiration has to be long enough that the exact distribution of extreme returns can be safely replaced by the asymptotic distribution. Practically, a Sherman goodness-of-fit test can be used to assess the convergence of the asymptotic distribution.[12]

A closed-form solution for the price of a crash option is

$$e^{-rT} NV \left\{ -\frac{\alpha_T^*}{-\tau^*} \Gamma\left[\tau^* + 1, \left(1 - \tau^* \frac{\beta_T^* - k}{\alpha_T^*}\right)^{1/\tau^*}\right] + \left[1 - F_z\left(\frac{\beta_T^* - k}{\alpha_T^*}\right)\right]\left[\frac{\alpha_T^*}{\tau^*} - \beta_T^* - k\right]\right\} \quad (16)$$

where the function Γ is defined by $\Gamma(x,y) = \int_0^y u^{x-1} e^{-u} du$, with $x > 0$, and the function F_Z is given by Equation (5).

Application to portfolio management

This section shows how crash options could be used in portfolio management. First, investment strategies are described. Then results based on simulations are presented.

Description of investment strategies

Three different investment strategies are considered: a long position, a long position protected by a put option and a long position protected by a crash option. The options are bought at the beginning of the investment period, and their maturity matches the investment horizon. The put option is taken at-the-money such that the initial value of the investment is fully protected (guarantee of capital). It means that the position is protected against any drop in the position value over the whole investment period. The crash option is taken with a strike of 0 per cent. A crash option does not offer a full protection but only insures the position value against the largest drop in value on a single trading period. In the simulation study, the position is assumed to be invested in the stock index. The frequency of the crash option is taken to be one trading day as a stock market crash (by definition) occurs on a short time-period. Four different investment horizons are considered: 1 month, 1 quarter, 1 semester and 1 year.

The strategies are tested over the time-period January 1885–December 1999 for the US stock market (Schwert database) and over the January 1992–January 2001 for the European stock market (Bloomberg database). Both databases are representative indexes of the equity market of each country. For example, for the European equity market, as of 31st January, 2001, it includes the following stocks: ABN Amro Holding NV, Aegon NV, Alcatel A, Allianz ag-reg, Astrazeneca plc, AXA, Banco Bilbao Vizcaya Argenta, Barclays plc, Bayer ag, BNP Paribas, BP Amoco plc, British Telecom plc, Banco Santander Central Hisp, Carrefour SA, CGNU plc, Credit Suisse group-reg, Daimlerchrysler ag-reg, Deutsche Bank ag-reg, Deutsche Telekom ag-reg, Diageo plc, E.on ag, ENI spa, Ericsson lm-b shs, France Telecom SA, Assicurazioni Generali, GlaxoSmithKline plc, HSBC holdings plc, ING groep n.v., L'OREAL, Lloyds TSB group plc, Marconi plc, Muenchener Rueckver ag-reg, Nestle SA-registered, Nokia oyj, Novartis ag-reg shs, Philips Electronics NV, Prudential plc, Roche holding ag-genusss, Royal Bank of Scotland group, Royal Dutch Petroleum, Shell Transport & Trading co plc, Siemens ag, Swiss re-reg, Telecom Italia spa, Telefonica SA, Total Fina Elf SA, UBS ag-registered, Vivendi Universal, Vodafone group plc, Zurich Financial Services.

For each market, the longest time-period is used in order to observe several stock market crashes. Let us take the US market for example. For the long position, the returns are computed over each non-overlapping periods of the whole time-period January 1885–December 1999. For example, considering yearly investment, the strategies are tested over 115 years. For the long position protected either by a put option or a cash option, the returns are computed without taking into account the option price (payoff of the strategy) and taking into account the option price (profit and loss of the strategy). The options' prices are obtained using a GARCH(1,1) process to model returns. By this method, option prices depend on the level of volatility at the beginning of each investment time-period. Note that returns taking into account the initial option price are subject to model risk and that returns not taking into account the initial options' prices are independent of the model used to price the options.

Results from simulations

Results are given in Table 5 for the US stock market and Table 6 for the European market. The following basic statistics are computed: mean, standard deviation, skewness and kurtosis. The worst-case scenario and the value at risk (VaR) at 95 per cent and 99 per cent are also reproduced. Let us

Table 5 Basic statistics of the investment strategies for the US stock market

	Long position	Long position + put option		Long position + crash option	
		Payoff	P&L	Payoff	P&L
		A. Investment horizon: 1 month			
Mean	0.76	2.25	0.50	2.46	0.48
Standard deviation	5.13	3.09	4.03	4.65	5.05
Skewness	−0.47	3.18	−3.44	0.52	−3.20
Kurtosis	7.46	25.38	47.88	9.22	32.92
Worst case	−32.79	0.00	−56.78	−23.68	−66.66
VaR (99%)	−15.36	0.00	−12.24	−11.07	−19.45
VaR (95%)	−7.21	0.00	−3.74	− 4.39	−7.14
		B. Investment horizon: 1 quarter			
Mean	2.36	4.62	1.62	4.83	2.14
Standard deviation	9.38	6.01	7.02	8.91	9.26
Skewness	−0.34	3.34	−2.65	0.51	−1.25
Kurtosis	8.11	22.97	23.52	10.01	7.33
Worst case	−45.84	0.00	−66.62	−38.10	−52.01
VaR (99%)	−30.92	0.00	−22.07	−25.06	−32.61
VaR (95%)	−11.50	0.00	−4.63	− 8.24	−12.47
		C. Investment horizon: 1 semester			
Mean	4.74	7.75	4.07	7.97	4.75
Standard deviation	13.02	8.51	9.74	12.53	13.51
Skewness	−0.66	1.26	−0.97	−0.38	−1.06
Kurtosis	2.73	1.73	9.23	3.04	4.56
Worst case	−61.02	0.00	−61.79	−55.33	−67.55
VaR (99%)	−30.39	0.00	−23.86	−22.20	−46.75
VaR (95%)	−18.86	0.00	−4.99	−12.78	−19.20
		D. Investment horizon: 1 year			
Mean	9.47	13.13	7.69	13.40	9.50
Standard deviation	17.88	12.64	14.37	17.46	18.09
Skewness	−0.45	0.56	−0.37	−0.29	−0.49
Kurtosis	0.02	−0.89	1.71	−0.16	0.06
Worst case	−44.40	0.00	−53.16	−36.07	−41.11
VaR (99%)	−44.40	0.00	−53.16	−36.07	−41.11
VaR (95%)	−18.66	0.00	−10.80	−14.90	−29.25

Note: This table gives the basic statistics for three different investment strategies (a long position, a long position protected with a put option and a long position protected with a crash option) and different investment horizons (from 1 month to 1 year). For the two insured strategies, both the payoff and the profits and losses (P&L) are given. Prices for put options and crash options are obtained by simulation by assuming a GARCH process for returns. The results are obtained for the US equity market over the period 1885–1999.

consider a yearly investment in the US stock market for example (Panel D of Table 5). The mean return of a yearly investment is equal to 9.46 per cent with a standard deviation of 17.88 per cent. The worst yearly return observed over the entire time period January 1885–December 1999 is equal to −44.40 per cent in year 1932. The mean return of a yearly investment protected with a put option is equal to 13.13 per cent (payoff) and 7.69 per cent (P&L) with a standard deviation of 12.64 per cent (payoff) and 14.37 (P&L). The worst case return is equal to 0 per cent (payoff) and −53.16 per cent (P&L) in year 1933. The mean return of a yearly investment protected with a crash option is equal to 13.39 per cent (payoff) and 9.51 per cent (P&L) with a standard deviation of 17.45 per cent (payoff) and 18.10 (P&L). The worst case return is equal to −36.07 per cent (payoff) and −41.11 per cent (P&L) in year 1932.

A put option and a crash option lead to different payoff distributions and then to different risk/return profiles. While a put option makes the payoff distribution completely asymmetric, a crash option by

Table 6 Basic statistics of the investment strategies for the European stock market

		[illegible]		[illegible]	
	Long position	Payoff	P&L	Payoff	P&L
A. Investment horizon: 1 month					
Mean	1.47	2.54	0.82	3.32	1.12
Standard deviation	4.34	2.78	2.85	[illegible]	[illegible]
Skewness	−0.51	1.18	0.24	−0.20	−0.77
Kurtosis	0.65	[illegible]	0.40	0.49	[illegible]
Worst case	−11.00	0.00	−7.79	[illegible]	[illegible]
VaR (99%)	−10.34	[illegible]	−4.44	−7.87	[illegible]
VaR (95%)	−7.67	0.00	−3.99	−3.63	−5.71
B. Investment horizon: 1 quarter					
Mean	4.82	5.75	2.77	7.37	4.80
Standard deviation	7.07	5.97	6.15	6.97	[illegible]
Skewness	0.47	1.05	0.87	[illegible]	[illegible]
Kurtosis	−0.32	0.52	−0.13	[illegible]	[illegible]
Worst case	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
VaR (99%)	[illegible]	[illegible]	−5.39	[illegible]	[illegible]
VaR (95%)	−6.83	0.00	−4.15	−2.22	[illegible]

Note: This table gives the basic statistics for three different investment strategies (a long position, a long position protected with a put option and a long position protected with a crash option) and different *investment horizons (from 1 month to 1 year). For the two insured strategies, both the payoff and the profits* and losses (P&L) are given. Prices for put options and crash options are obtained by simulation by assuming a GARCH process for returns. The results are obtained for the European equity market over the period 1992–2001.

removing the worst realisation of returns over a single trading period more or less translates the payoff distribution to the right without changing its shape.

Conclusion

This paper focuses on portfolio management during the most volatile periods. First, a quantitative characterisation of stock market crashes using extreme value theory is proposed. Then a new type of option that could enhance the management of financial assets during periods of high volatility is introduced. A simulation study is finally carried out to test investment strategies using put options and crash options.

Crash options may improve the performance of portfolio insurance techniques, which work badly during periods of market stress. Although a given framework for the trading of crash options is suggested, potential users may have a different opinion concerning the choice of the underlying interest, the definition of the payoff, the type of exercise, the settlement procedure. Crash options as defined in this paper deal with the largest fall in the asset price; similar options may consider the second, the third, the *n*th largest falls in asset prices; combinations of such crash options may be traded as well. There is also strong evidence that the process of financial asset prices is heteroscedastic, and this has a direct influence on the joint distribution of positive and negative extreme price changes as explained in Longin (1993); a package including both a boom option and a crash option may then be of some interest for investors.

This paper first deals with the hedging and pricing of crash options in a perfect, continuous Gaussian market. The assumption of normality, however, leads to an underestimation of the weight of the distribution tails which are central to the pricing of such options. Reality may be better described with a Fréchet market characterised by large price movements. An asymptotic pricing formula based on extreme value theory is then derived to take into account the right amount of extremes.

Further research may consider a particular feature observed in reality that is not addressed in this paper: extreme price movements are usually associated with discontinuities in the price and trading processes. For example, the biggest stock market crashes resulted from price jumps, while the market was closed by the Exchange for an unspecified time. This feature may make boom and crash options look even more attractive instruments for buyers as they complete the market, but also more difficult to hedge for issuers during periods of extreme volatility. Such a difficulty may be overcome with the special institutional arrangement proposed below.

As crash options appear to have certain similarities to insurance products, the writing of such contracts should be left to big financial institutions. A special role may also be given to the Central Bank, which may act as a reinsurance company for these financial institutions acting as insurance companies. Similarly to the market for PCS Catastrophe Insurance Options organised by the Chicago Board of Trade (1995), financial institutions may build crash option spreads. For example, a financial institution writing crash options for a customer with a striking return of −3 per cent could limit its exposure to the very large stock market crashes like 1929 and 1987 by buying at the same time a crash option with a striking return of −10 per cent from the Central Bank taking the ultimate risk. As the hedge portfolio composed of the underlying interest and the risk-free bond may poorly replicate boom and crash options during the most volatile periods, such an institutional arrangement may help to solve the hedging problem. Note that such an activity of the Central Bank may rationalise its role of lender of last resort, sometimes undertaken during stock market crashes. This paper would then provide the market price that the Central Bank should charge for such a service.

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Notes

1 The stock market crash of October 1987 also highlighted the importance of transaction costs as explained by Rubinstein (1988) and the functioning of financial markets as emphasised by the US Commodity Futures Trading Commission report (1987: 55–61).

2 The Brady report (1988) partly blames programme trading (portfolio insurance and index arbitrage) for the stock market crash of October 1987.

3 See the *Wall Street Journal* (17th October, 1988).

4 Similarly, boom options could be introduced to protect the value of a short position against a sharp, large rise in market prices (see Longin, 1996b).

5 See Goldman *et al.* (1979) and Conze and Viswanathan (1991) for a presentation of lookback options and their pricing.

6 The *New York Times* is the only daily newspaper which entirely covers the period 1885–1999.

7 A futures index may be preferred to a stock index because of the problems in the stock market during highly volatile periods (lack of liquidity and informationless prices resulting from trading halts in particular stocks and the difficulty of getting on-time stock prices).

8 See Longin (1995) for a method based on extreme price movements to set margin levels in derivative markets.

9 An increase in S leads to an increase in the probability of a higher striking price because of a higher value of Min $[k, Z_t(f)]$.

10 Note that, in the classical framework, it would be optimal to exercise an American crash option just before the crash of 19th October, 1987! As shown in Table 3, after the large price decrease on 16th October, it is more profitable to invest the sure value of the crash option in a risk-free bond than to hold the crash option expecting further gains. The reason is that in a Gaussian market, a price decrease larger than the one observed on 16th October (−5.16 per cent) is very unlikely.

11 Such an assertion comes directly from the formulae relating the parameters of the asymptotic extreme value distribution to the parameters of the basic process as given in Leadbetter *et al.* (1983: 20–1).

12 Longin (1996) finds that the asymptotic distribution of extreme returns selected over a period longer than one semester describes very well the behaviour of observed extreme returns. The longer the time to expiration of boom options, the more accurate the asymptotic pricing formula. Numerical values show that the pricing error is small: for example, in the case of normality, the price of a boom option with a striking return of 0 per cent and a maturity of one year is $28,425.49 using the exact distribution of maximal returns and $28,805.29 using the asymptotic Gumbel distribution, a percentage difference around 1 per cent. Boom options with a short maturity may be difficult to price with great precision. The trading of such options, however, would be likely to be small, as investors may roll over their positions as time goes on.

References

Berman, S. M. (1964) 'Limiting Theorems for the Maximum Term in Stationary Sequences', *Annals of Mathematical Statistics*, 35, 502–16.

Black, F. and Scholes, M. (1973) 'The Pricing of Options and Corporate Liabilities', *Journal of Political Economy*, 81, 637–59.

Bollerslev, T. (1986) 'Generalized Autoregressive Conditional Heteroskedasticity', *Journal of Econometrics*, 31, 307–27.

Boulier, J.-F., Delaud, R. and Longin, F. M. (1998) 'Application de la Théorie des Valeurs Extrêmes aux Marchés Financiers', *Banques et Marchés*, 32, 5–14.

Boulier, J.-F. and Sikorav, J. (1992) 'Portfolio Insurance: The Attraction of Security', *Department of Research and Innovation, CCF Group, Quants Review*, 6.

Brady, N. (1988) *Report of the Presidential Task Force on Market Mechanisms*.

Chicago Board of Trade (1995) *PCS Options. A User's Guide*, Chicago.

Conze, A. and Viswanathan, (1991) 'Path Dependent Options: The Case of Lookback Options', *Journal of Finance*, 46, 1893–907.

Cox, J. C. and Rubinstein, M. (1985) *Options Markets*, Prentice Hall, Englewood Cliffs, NJ.

De Haan L., Resnick, I. S., Rootzn, H. and De Vries, C. G. (1989) 'Extremal Behavior of Solutions to a Stochastic Difference Equation with Applications to ARCH Process', *Stochastic Processes and their Applications*, 32, 213–24.

Dekkers, A. L. M. and De Haan, L. (1989) 'On the Estimation of the Extreme Value Index and Large Quantile Estimation', *Annals of Statistics*, 17, 1795–832.

Duan, J.-C. (1995) 'The GARCH Option Pricing Model', *Mathematical Finance*, 5, 13–32.

Engle, R. F. (1982) 'Auto-regressive Conditional Heteroskedasticity with Estimates of the Variance of United Kingdom Inflation', *Econometrica*, 50, 987–1007.

Fama, E. (1963) 'Mandelbrot and the Stable Paretian Hypothesis', *Journal of Business*, 36, 420–9.

Feller, W. (1971) *An Introduction to Probability Theory and its Applications*, Vol. 2, Wiley, New York.

Galambos, J. (1978) *The Asymptotic Theory of Extreme Order Statistics*, John Wiley, New York.

Gnedenko, B. V. (1943) 'Sur la Distribution Limite du Terme Maximum d'une Série Aléatoire', *Annals of Mathematics*, 44, 423–53.

Goldman, B. M., Sosin, H. B. and Gatto, M. A. (1979) 'Path Dependent Options: "Buy at the Low, Sell at the High"', *Journal of Finance*, 34, 1111–27.

Gumbel, E. J. (1958) *Statistics of Extremes*, Columbia University Press, New York.

Harrison, J. M. and Kreps, D. M. (1981) 'Martingales and Arbitrage in Multiperiod Securities Markets', *Journal of Economic Theory*, 20, 381–408.

Jenkinson, A. F. (1955) 'The Frequency Distribution of the Annual Maximum (or Minimum) Values of Meteorological Elements', *Quarterly Journal of the Royal Meteorology Society*, 87, 145–58.

Kindleberger, C. P. (1978) *Manias, Panics and Crashes*, 2nd edn, Macmillan, Basingstoke.

Kon, S. (1984) 'Models of Stock Returns. A Comparison', *Journal of Finance*, 39, 147–65.

Leadbetter, M. R., Lindgren, G. and Rootzèn, H. (1983) *Extremes and Related Properties of Random Sequences and Processes*, Springer, New York.

Longin, F. M. (1993) 'Volatility and Extreme Movements in Equity Markets', PhD thesis, HEC, France.

Longin, F. M. (1995) 'Optimal Margins in Futures Markets: A Parametric Extreme-based Approach', *Proceedings of the Seventh CBOT Conference on Futures and Options*, Bonn, Germany.

Longin F. M. (1996a) 'The Asymptotic Distribution of Extreme Stock Returns', *Journal of Business*, 63, 383–408.

Longin, F. M. (1996b) 'Winning in the Best and Worst of Times: Boom and Crash Options', CERESSEC Working Paper 96-005, ESSEC.

Longin, F. M. (1997) 'The Threshold Effect in Expected Volatility: A Model Based on Asymmetric Information', *The Review of Financial Studies*, 10, 837–69.

Longin, F. M. (2000) 'From VaR to Stress Testing: The Extreme Value Approach, *Journal of Banking and Finance*, 24, 1097–130.

Longin, F. M. and Solnik, B. (1995) 'Is the Correlation in International Equity Returns Constant: 1960-1990?' *Journal of International Money and Finance*, 14, 3–26.

Longin, F. M. and Solnik, B. (2001) 'Extreme Correlation of International Equity Markets', *The Journal of Finance*, 56, 651–78.

Mandelbrot, B. (1963) 'The Variation of Certain Speculative Prices', *Journal of Business*, 36, 394–419.

McCulloch, J. H. (1978) 'Continuous Time Processes with Stable Increment', *Journal of Business*, 51, 601–19.

Praetz, P. D. (1972) 'The Distribution of Share Price Changes', *Journal of Business*, 45, 49–55.
Press, S. J. (1967) 'A Compound Events Model for Security Prices', *Journal of Business*, 40, 317–35.
Rubinstein, M. (1988) 'Portfolio Insurance and the Market Crash', *Financial Analysts Journal*, 43, 38–47.
Rubinstein, M. and Leland, H. E. (1981) 'Replicating options with Positions in Stock and Cash', *Financial Analysts Journal*, 37, 63–72.
Schwert, W. G. (1990) 'Indexes of US Stock Prices from 1802 to 1987', *Journal of Business*, 63, 399–427.
US Commodity Futures Trading Commission (1987) *Interim Report on Stock Index Futures and Crash Market Activity during October 1987*, Division of Economic Analysis and Division of Trading and Markets.
Wall Street Journal (1988) 'Changes Since Crash in US Can't Prevent a Repeat Performance', 17th October.