TERM CAPITAL-GUARANTEED FUND MANAGEMENT : THE OPTION METHOD VS THE CUSHION METHOD

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Abstract

Term capital-guaranteed funds are managed so that the investor recovers at maturity their initial capital, the fund performance being related to the performance of financial markets. The aim of this paper is to investigate two types of fund management, namely the option method and the cushion method. In the first case, the fund manager statically hedges the fund using options. In the second case, the fund manager dynamically allocates the wealth following specific trading rules to insure the fund will fulfil the guarantee. For both types of management we describe the final value of the fund, we illustrate the fund behaviour for typical market evolutions and we study the distribution of fund values at maturity. Finally, we analyse the risk and performance characteristics of the fund with various measures and discuss optimality. Our results may help fund managers to choose the adequate fund management method.

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INTRODUCTION

Many investors would like to receive the gains during bullish markets without bearing the losses during bearish markets. Among the different financial products currently available to investors, such an objective is achieved by so-called guaranteed funds, which insure investors to get back the initial value of their investment. These financial products are especially popular after a market downturn when investors have directly invested in financial markets and suffer heavy losses.

Guaranteed funds provide to the holder the guarantee to recover the totality of her initial capital, sometimes increased by an extra profit related to the out-performance of financial markets. The “level of the guarantee” is called the floor value of the fund at maturity. The present paper focuses on “term capital-guaranteed funds”, in the sense that the floor applies at fund maturity only. The guarantee is therefore supposed to be “European” versus “American”. The former applies at the maturity, while the latter applies at any time before maturity.

Given the growing importance of guaranteed funds, a practical question in the industry is the choice of the management method to fulfil the guarantee. Two methods are used in practice: the option method and the cushion method. In the case of the option method developped by Leland and Rubinstein (1976), the fund manager statically hedges the fund using options. In the case of the cushion method developped by Perold and Sharpe (1988), the fund manager dynamically allocates the wealth following specific trading rules to insure the fund will fulfil the guarantee. In this paper we ask several questions: how is the fund value related to the performance of financial markets? What are the similarities and differences between the two methods? Are these methods equivalent or is one better than the other?

It is often said that the main advantage of the cushion method is its flexibility over time in terms of the choice of the underlying asset and degree of riskiness while the main advantage of the option method is its low managing cost as it is structured once in all and the possibility to communicate on the known fund performance relative to the market performance. In this paper we focus on the risk and performance issues.

The goal of this paper is to present and compare the option method and the cushion method in a rigorous way as there are few works done by academics or practitioners that consider the two methods at the same time. The results obtained in this paper may help practitioners to choose the management method for their funds. The comparison is done within a standard framework: a continuous-time Brownian motion process for the risky asset with constant interest rate, constant risk premium and constant volatility.

This paper is organised as follows : the first and second sections describe in detail each fund management technique. The third section then provides results based on simulations in order to illustrate each method for particular market evolutions and to study the distribution of the final value of the fund managed by each technique. The final section deals with the risk/performance profile. Using different measures of risk and performance used by practitioners, the optimality of the methods is discussed. The conclusion summarises our results and relates them to the academic literature based on the concept of utility.

1. FUND MANAGED WITH THE OPTION METHOD

The option method was first introduced by Leland and Rubinstein (1976). It is known as the OBPI (Option Based Portfolio Insurance) method. With this method, the fund is structured at the beginning in such a way that the difference between the initial fund value and the discounted floor value, is invested in options on a chosen underlying asset while a risk-free zero-coupon bond with a nominal equal to the floor value is bought to fulfil the guarantee at maturity. The options can either be bought in the derivatives market, or synthetically replicated following a hedging strategy.

1.1 Fund structuration

We consider two optional structures: a standard call option and capped call options. Capped call options allow one to increase the profit due to average or good market performances by discarding the profits due to exceptional performances. Note that a standard call option corresponds to a particular capped call option with a cap value equal to infinity. Depending on the initial level of implied volatility and interest rates, it might not be possible to buy an option with a nominal equal to the initial value of the fund but to a fraction of this value only. This parameter called the "gearing" of the fund specifies the nominal amount of the underlying asset, on which the option is written. Note that the strategy might also be described as a long position on a fraction λ of the initial fund value combined with the buying of a put option (see El Karoui *et al* (2002)).

In the case of a standard call option, the option value at maturity T is given by:

(1) $$C_T^{\infty} = max(\lambda_{\infty} \cdot S_T - K; 0),$$

where S denotes the value of the underlying asset, K the strike of the option equal to the guarantee of the fund at maturity, λ the gearing of the fund and the subscript ∞ stands for a call capped at infinity.

In the case of a capped call option with a cap value equal to K', the option value at maturity T is given by :

(2) $$C_T^{K'} = max(min(\lambda_{K'} \cdot S_T - K, K' - K); 0),$$

where K' is the maximum extra profit beyond the capital guarantee K.

In all cases, the gearing parameter λ must be adjusted at the initial launching date so that the initial value of the fund equals the value of the zero-coupon bond added to the value of the option:

(3) $$V_0 = K \cdot exp(-r \cdot T) + C_0^{K'}(\lambda_{K'} \cdot S_0, K, T),$$

where $C_0^{K'}(\lambda_{K'} \cdot S_0, K, T)$ is the value at time 0 of a call option with nominal amount $\lambda_{K'} \cdot S_0$, exercise price K, cap K' (K' being possibly equal to $+\infty$) and maturity T.

Table 1 gives the value of the gearing parameter for a standard call option and for capped call options. The higher the cap value K', the lower the gearing value λ due to the decreasing relationship between the capped call value and the cap value.

1.2 Fund valuation with the option method

While the fund value is known at the initial and final dates without ambiguity, this is not the case at intermediate dates. A valuation model must be developed to solve this problem. We present below a standard model.

The risk-neutral implied dynamics of the risky asset price S is classically given by the following Black-Scholes stochastic differential equation :

$$(4) \qquad \frac{dS_t}{S_t} = r \cdot dt + \sigma \cdot dW_t \,,$$

where r is the risk-free rate, σ the volatility, both assumed to be constant, and $(W_t)_{t \geq 0}$ is a standard Brownian motion under the risk-neutral probability.

At any time t before maturity T, the fund value denoted by V_t is given by :

$$(5) \qquad V_t = K \cdot exp(-r \cdot (T - t)) + C_t^{K'}(\lambda_{K'} \cdot S_t, K, T),$$

where $C_t^{K'}(\lambda_{K'} \cdot S_t, K, T)$ is given by a Black-Scholes formula (see Appendix 1).

1.3 Final fund value with the option method

Figure 1 represents the final fund value as a function of the risky asset price in the case of the option method. The fund value is simply equal to the sum of the floor value and the option payoff given by Equations (1) or (2).

2. FUND MANAGED WITH THE CUSHION METHOD

The cushion method was first introduced by Perold and Sharpe (1988). The method is known as the CPPI (Constant Proportion Portfolio Insurance) method. It consists in defining dynamically a self-financing trading strategy in a risky asset (or a combination of risky assets). All through the paper, we consider that the fund manager has already chosen what is usually called the "tactical allocation", namely the composition of the risky portfolio among traded assets.[3] We therefore focus on the problem of defining the "strategic allocation", namely the management of the investment in risky assets compatible with the obligation to fulfil the guarantee.

2.1 Dynamic strategies

Within the cushion method, we consider both constrained and unconstrained investment strategies. The constraint deals with the maximum allowed in the risky asset.

2.1.1 Unconstrained strategies

At any time t, the fund value is decomposed in two parts: the discounted floor value and the cushion:

$$(6) \qquad V_t = K \cdot exp(-r \cdot (T - t)) + C_t \,,$$

where the cushion, denoted by C_t, is obtained as the difference between the fund value and the discounted floor value. With this method, a part of the fund is invested in a risk-free zero-coupon bond maturing at time T and another part in a risky asset. Note that according to the importance of the part invested in the risky asset relative to the fund value, the investment in the bond can be long or short (when the investment in the risky asset exceeds the fund value). A classical investment strategy is the following :

[3] See Brennan *et al* (1997) for a detailed presentation of this issue.

the amount invested at time *t* in the risky asset of value S_t, denoted by $\phi_t^m \cdot S_t$, is set equal to a multiple *m* of the cushion value:

$$(7) \qquad \phi_t^m \cdot S_t = m \cdot C_t .$$

The multiple parameter *m* is usually called the « leverage » of the fund. When *m* is equal to 1, we invest the cushion only. When *m* is greater than 1, we invest a much bigger amount in the risky asset which is why this type of fund is said to be leveraged.

2.1.2 Constrained strategies

Additional constraints might be added to the CPPI trading rule. For example, a maximum can be specified for the proportion of the fund invested in the risky asset. A usual constraint is that the value invested in the risky asset must not exceed the fund value itself, hence preventing any borrowing.

Let us denote by *b* the maximum proportion of current fund value invested in the risky asset. Note that the constrained strategies preventing any borrowing correspond to the cases: $b \leq 100\%$. and that the unconstrained strategy presented above corresponds to the special case: $b=+\infty$. The amount invested at time *t* in the risky asset, denoted by $\phi_t^{m,b} \cdot S_t$, is set equal to the minimum between the unconstrained strategy and the maximum allowed in the risky asset:

$$(8) \qquad \phi_t^{m,b} \cdot S_t = min(m \cdot C_t , b \cdot V_t).$$

2.2 Fund valuation with the cushion method

Following the trading strategy described above, at any time *t*, the fund is decomposed in two parts: the zero-coupon bond and the risky asset:

$$(9) \qquad V_t = B_t + \phi_t^{m,b} \cdot S_t ,$$

As the values of the zero-coupon bond and of the risky asset can be directly obtained from market prices, the fund value is straightforward.

Remembering that $V_t = K \cdot exp(-r \cdot (T-t)) + C_t$ and the lemma given in Appendix 2 yields to the following risk-neutral stochastic differential equation for $C_t^{m,b}$:

$$(10) \qquad dC_t^{m,b} = r \cdot C_t^{m,b} \cdot dt + min\left(m \cdot C_t^{m,b},\ b \cdot \left(K \cdot exp(-r \cdot (T-t)) + C_t^{m,b}\right)\right) \cdot \sigma \cdot dW_t .$$

Integrating explicitly Equation (10) is not an easy task, but V_T can be easily exhibited through numerical simulations as shown in the following subsection. Only in the special case of the unconstrained dynamic strategy, the final fund value can be explicitly written as an exponential function of the risky asset price at fund maturity (see Appendix 3), which proves that it is path-independent. In the general case with the constraint ($b<+\infty$) path-dependent features appear in the solution.

2.3 Final fund value with the cushion method

Figure 2 represents the final fund value in the case of the cushion method.

As mentioned above, for the unconstrained investment strategy, the final fund value has an exponential form. The main impact of the additional investment constraint is that this exponential form disappears, the behaviour of the fund tending to be logarithmic for large values of the risky asset price at fund maturity. By imposing this constraint, we exchange high performances of the fund with low and medium performances. This result can be deduced intuitively by looking carefully at Equation (10). Due to the minimum in the variance term of Equation (10), the final distribution of $C_t^{m,b}$ is a mixture of two regimes, which are distinct from each other for respectively low and high values of the cushion. For low values of $C_t^{m,b}$, the minimum gives $\phi_t^{m,b} \cdot S_t = m \cdot C_t^{m,b}$, which makes the strategy equivalent to the unconstrained one. For high values of $C_t^{m,b}$, the minimum gives $\phi_t^{m,b} \cdot S_t = b \cdot \left(K \cdot exp\left(-r \cdot (T-t)\right) + C_t^{m,b}\right)$, with a predominance of the term $b \cdot C_t^{m,b}$ at infinity. When the value of parameter *b* is less than 100%, the constrained strategies have a logarithmic behaviour as observed in Figure 2. Following this argument, the special case *b*=100% gives a close to a linear behaviour of the final fund value. This feature makes the dynamic strategy possibly close to an option-based strategy.

3. FUND BEHAVIOUR

This section compares the behaviour of guaranteed funds managed with the option method and of funds managed with the cushion method. First, the fund behaviour is studied for typical market evolutions: a bearish market and a bullish market. Then, the statistical distribution of the final fund value is obtained. Finally, basic statistics are computed to summarise the fund behaviour.[4]

3.1 Fund value over time

In order to consider the fund value in the future, we need to specify the dynamics of the risky asset price under the historical probability. In order to allow for a description of returns under the historical probability and to discuss optimality, a risk-premium is added to Equation (1) making the evolution of $(S_t)_{t\geq 0}$ under the historical probability:

$$(11) \qquad \frac{dS_t}{S_t} = (r + \Lambda \cdot \sigma) \cdot dt + \sigma \cdot dW_t,$$

where price of risk Λ is set equal to 0.25, which is implying an expected annual return under the historical probability of 10%.

Practitioners pay a particular attention to the evolution of the fund value over its lifetime. Indeed, the fund value at any time has to be higher than the zero-coupon bond paying the floor value at maturity.

The evolution of the fund value in a bullish market is represented in Figure 3A (option method) and Figure 3B (cushion method). Symmetrically, Figures 4A and 4B deal with a bearish market. In the case of a bullish market, fund performances are inferior to the performance of the underlying risky asset due to the cost of the insurance. In the case of a bearish market, the insurance implies the opposite : the final value is above the level of guarantee despite a market fall beyond this level.

With both methods, current values for intermediary dates are superior to the zero-coupon bond price $K \cdot exp(-r \cdot (T-t))$ but not necessarily superior to the guarantee level *K* as the guarantee being European applies only at maturity. This is illustrated in Figures 4A and B.

Distinctive behaviours are observed for constrained and unconstrained strategies. In both option based and dynamic strategies, the constrained strategies (with *K'*=20% and *b*=20%) outperform the others in the case of low or medium performance of the risky asset but underperform in the case of high performance.

Comparing the two types of management lead to the following remark : for low performances of the stock market, the cushion method gives higher returns than the option method. For medium performances, the option method gives higher returns than the cushion method. And finally for high performances, the cushion method outperforms the option method again. This result is detailed in the following subsections.

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[4] Another interesting approach as developped by Bertrand and Prigent (2002) would be to consider the OBPI in the CPPI framework (what is the equivalent dynamic strategy or the value of the leverage parameter *m*?) and conversely to consider the CPPI in the OBPI framework (what is the equivalent hedging strategy or the value of the hedge parameter Δ?)

3.2 Distribution of the final fund value

The distribution of the final fund value is represented in Figure 5A (option method) and Figure 5B (cushion method). The distributions obtained with the option method are truncated log-normal. The high peaks in the distribution are explained by the saturation of the constraints imposed on the final fund value (either the floor value at K or the cap values at K'). The distributions obtained with the cushion method may present one or two peaks according to the level of investment constraint. For the unconstrained strategy ($b=+\infty$), the log-normal distribution gives a maximal frequency for low values (final value around 102% of the initial value), meaning that the main features of the unconstrained strategy are : high frequency for low performances, relatively low frequency for medium performances, and fat tails for high performances. For constrained strategies (the maximum invested in the risky asset b being lower than 100%), the mixture of distributions described in the previous section is visible: the distribution tends to have two modes, the first one being related to low performances as in the unconstrained case, the second one being displaced to average performances of the stock market (around 110 % for b=20 % and around 120% for b=40%). These two modes correspond to the two regimes described in subsection 2.3.

3.3 Statistics

Table 2 gives descriptive statistics about the fund returns under the simulated historical probability. The mean and standard deviation of the final payoffs behave as intuition expects : for both the option method and the cushion method, increasing K' or b implies an extra expected return together with an extra standard deviation. The standard deviation of the funds is generally lower than for the risky asset due to the guarantee, except in the case of the unconstrained cushion strategy due to the leverage effect. The negative value of the kurtosis for funds managed with the option methods with capped call values K' ranging from 20% to 60% is due to the flatness of the truncated distributions. On the other hand, the positive value of kurtosis for funds managed with the cushion method with investment constraint values ranging from 80 % to $+\infty$ is due to the exponential form of the final payoff function described in Figure 2. Due to the fatter tails of the distributions obtained with the cushion method one could expect to have higher top quantiles than for funds managed with the option method. Noticeably enough, the top 5% quantile values remain higher for funds managed with the option method. However, the top 1% quantile values give the expected result in favour of the cushion method. This means that one has to expect extreme performances of the stock market in order to take advantage of the relative distribution of funds managed with the cushion method.

The statistical results described above imply that the main difference between the two types of management is related to the importance of the distribution tails. These comparative results lead to a natural question : what is the best choice between the two types of fund management? In the following section we describe how the answer to this question depends on the way risk and performance are measured.

4. RISK AND PERFORMANCE MEASURES AND OPTIMALITY

In this section, we follow the classical idea of Markowitz (1959) further extended by Merton (1990) that a manager either maximises performance for a given level of risk or minimises risk for a given level of performance. On a risk-performance graph, the manager therefore tends to optimally choose strategies giving a couple located on the left-hand side and upper side of the graph. We introduce various measures of risk and performance in order to discuss optimality.

4.1 Standard risk and performance measures

As a benchmark for further discussion, standard measures are used to describe fund returns: the mean for performance and the standard deviation for risk. Figure 6 plots the risk-performance couples for funds managed with the option method and for funds managed with the cushion method. For both methods, a higher mean is associated with a higher standard deviation. However, the option method clearly appears to be more efficient than the cushion method as, for a given risk level, the mean of the former method is in most cases higher than the mean of the latter method. Noticeably enough, for the option method, the mean tends to increase at a higher speed than for the cushion method. One could therefore conclude that the cushion method is less efficient except in the case of highly constrained strategies (when parameter *b* representing the maximum invested in the risky asset is constrained to be less than 20% or 30% of the fund value).

Still, results might highly depend on the way both risk and performance are measured. We have already noticed that the main difference between the two types of management is the statistical behaviour of the fund for extreme returns located in the right tail of the distribution. In the following subsections we consider alternative measures of risk and performance.

4.2 Alternative performance measure

Instead of looking at the mean of the return distribution, which is a global measure of performance, we consider top quantile measures (right tail of the distribution) focusing on the best performances of the fund. Choosing the quantile as a measure of performance implies giving a higher weight to extreme positive moves in asset prices.

Figure 7 plots the risk-performance couples, with performance measured by the top 5% quantile replacing the mean, and risk measured by the standard deviation. The cushion method is now located on the left of the graph making it more efficient than the option method for that choice of measures. This is due to the relatively fatter tails of the return distribution obtained with the cushion method. Replacing the top 5% quantile by the top 1% quantile heightens this phenomenon.

4.3 Alternative risk measure

In this subsection, we investigate an alternative risk measure based on Stone (1973). As explained by Stone, the choice of a risk measure implicitly involves decisions about: 1) a reference level of wealth about which deviations are measured; 2) the relative importance of small versus large deviations; and 3) the outcomes that should be included in the risk measure. Stone defines two related risk measures denoted by L and R:

$$(10) \qquad L(W_0,k,A)=\int_{-\infty}^{A}|W-W_0|^k\,dF(W) \text{ and } R(W_0,k,A)=\left(\int_{-\infty}^{A}|W-W_0|^k\,dF(W)\right)^{\frac{1}{k}},$$

This general formula depends on three parameters, which allows one to address the three issues mentioned above: W_0 (about what point are the deviations to be measured?), k (what is the relative importance of large deviations with respect to small deviations?) and A (which of the deviations are to be counted in specifying risk measure?). Note that this general formula encompasses various traditional risk measures. First, when $W_0=\overline{W}$, k=2 and A=+∞, the risk measure $L(\overline{W},2,+\infty)$ corresponds to the variance. Second, when $W_0=\overline{W}$, k=2 and $A=\overline{W}$, the risk measure $L(\overline{W},2,\overline{W})$ corresponds to the semi-variance. Third, when k=0 and $A=D$, the risk measure $L(W_0,0,D)$ corresponds to the probability of doing less than threshold D.

In our case, we choose $W_0=\overline{W}$ and A=+∞, and focus on the parameter k. When $1<k<+\infty$, large deviations assume relatively more importance than small deviations. When k=1, all deviations are weighted equally. When $0<k<1$, small deviations assume relatively more importance than large deviations. When k=0, we obtain a degenerate case in which only the probability of the event is considered. In our study, we consider the intermediate case k=1, for which the risk measure corresponds to the mean absolute deviation.

Figure 8 plots the risk-performance couples, with performance measured by the usual mean and risk measured by the mean absolute deviation replacing the standard deviation. Due to the lower impact of a large deviation on risk, the cushion method now appears to outperform the option method. Especially for the least constrained cushion strategies, the exponential behaviour of the strategy, which was previously considered as a drawback, now appears to become a positive feature making it possibly optimal.

One could obviously play with previous results and design a couple of measures, which would make dynamical strategies look highly more efficient than option based methods. One of our messages is therefore to focus on the importance of the choice of measures for risk and performance, especially as far as the relative impact of large deviations is concerned.

5. CONCLUDING REMARKS

In this paper, we have analysed two types of term capital-guaranteed fund management: the option method and the cushion method. Both methods allow one to achieve the main objective of this type of fund: to fulfil the guarantee at maturity. However, the two methods differ in many ways. The fund values at maturity are different for a given evolution of financial markets. As a consequence the statistical distributions of the fund value at maturity and the statistics summarising the risk/performance profile of the fund are also different.

The main results presented in this paper can be summarised as follows: First, funds managed with the cushion method exhibit a return distribution with fatter tails than funds managed with the option method, emphasising the importance of extreme returns. Second, the option method seems to dominate the cushion method in the classical mean-variance framework used to analyse the risk/performance characteristics of the funds. However, when alternative measures of performance and risk, such as top quantiles and the absolute deviation, are used, this result can be reversed.

Remains the fact that there is no robust theoretical evidence in favour of one specific fund management method or final payoff, which makes the job of optimally tailoring a fund still highly dependent on subjective parameters such as the parameters describing risk and performance discussed in the paper.

Along these lines, several authors have introduced the concept of utility to deal with optimality. Among those, it is noticeable that Black and Perold (1992) proved that CPPI strategies maximise the expected utility of the final wealth for well chosen piece-wise constant risk aversion utility functions. In their setting, the wealth constraint stating that the final floor is dealt by introducing a linear utility function for wealth values inferior to the floor. In a similar setting, it is easy to prove (see e.g. El Karoui et al, 2002) that CPPI maximises constant relative risk aversion functions, when utility applies on the extra wealth over the floor. Such a framework – considering the utility above the floor only - may be justified by the fact that the wealth becomes risky above that level. More generally several authors have solved the problem of maximising the expected utility under a final wealth constraint (see e.g. Cox and Huang, 1989, and El Karoui et al, 2002). Under very unrestrictive conditions on utility (among which strictly concavity), it is proven that the option method is always optimal when written on the portfolio which maximises the expeced utility with no constraint.

Being aware of the on-going academic discussion regarding the optimality of the cushion method versus the option method, we have intended in this paper to give a precise and complete description of both methods, in a setting as close as possible to the market practice.

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Appendix 1

Final fund value with the option method

The value of a capped call at time 0, denoted by $C_0^{K'}(\lambda_{K'} \cdot S_0, K, T)$, is given by the difference between two Black-Scholes formulae figuring two call options with respective strike values *K* and *K'*.

$$C_0^{K'}(\lambda_{K'} \cdot S_0, K, T) = S_0 N(d_{1,1}) - K\exp(-rT) N(d_{1,2}) - S_0 N(d_{2,1}) + K'\exp(-rT) N(d_{2,2}),$$

where $d_{1,1} = \dfrac{ln\dfrac{S_0}{K} + \left(r + \dfrac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and $d_{1,2} = d_{1,1} - \sigma\sqrt{T}$;

and $d_{2,1} = \dfrac{ln\dfrac{S_0}{K'} + \left(r + \dfrac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and $d_{2,2} = d_{2,1} - \sigma\sqrt{T}.$

Appendix 2
Final fund value with the cushion method

In order to describe the evolution of the current value of the cushion C_t, we recall the general property of self-financing trading strategies :

Lemma: A self-financing strategy with horizon date *T* is defined by the process $(\phi_t)_{0\leq t\leq T}$ figuring the nominal amount currently invested in the risky asset $(S_t)_{t\geq 0}$. Let us denote by π_t the current value of the strategy. The self-financing property implies that the risk-neutral dynamics of π_t is given by :

(1) $$d\pi_t = r\cdot\pi_t\cdot dt + \phi_t\cdot S_t\cdot\sigma\cdot dW_t$$

Proof: Due to the arbitrage free assumption the discounted value of a self-financing portfolio $\pi_t\cdot exp(-r\cdot t)$ is a martingale under the risk-neutral probability measure. The martingale representation theorem implies that there exists an adapted process $(\psi_t)_{0\leq t\leq T}$ such that :

(2) $$d\pi_t = r\cdot\pi_t\cdot dt + \psi_t\cdot dW_t$$

Considering that $(\phi_t)_{0\leq t\leq T}$ specifies the nominal amount invested in risky assets at time *t*, the variance term ψ_t equals $\phi_t\cdot S_t\cdot\sigma$.

Applying previous lemma with $\phi_t\cdot S_t$ given by Equation (8) in the text yields to Equation (10).

Appendix 3

Log-normality of the cushion method

Applying the lemma given in Appendix 2 to the classical CPPI method with leverage *m*, for which $\phi_t \cdot S_t = m \cdot C_t$ and $\pi_t = C_t$, we get the following risk-neutral stochastic differential equation for C_t:

(1) $$dC_t = r \cdot C_t \cdot dt + m \cdot C_t \cdot \sigma \cdot dW_t ,$$

with initial condition $C_0 = V_0 - K\, exp(-r \cdot T)$. This yields to the following corollary :

Corollary: The dynamic strategy with leverage *m* gives a log-normal final value for the cushion C_T (when continuously rebalanced). C_T equals is a *m*-exponential function of the final performance of the risky portfolio $\frac{S_T}{S_0}$:

Proof: Integrated Equation (3) gives :

(2) $$C_T = C_0 \cdot \exp\left(\left(r + \frac{1}{2} m^2 \sigma^2\right) \cdot T + m \cdot \sigma \cdot W_T\right),$$

which is log-normally distributed. Equation (4) can also be written using $S_T = S_0 \cdot exp\left(\left(r + \frac{1}{2}\sigma^2\right) \cdot T + \sigma \cdot W_T\right)$ following :

(3) $$C_T = C_0 \cdot \left(\frac{S_T}{S_0}\right)^m \cdot \exp\left((m-1) \cdot \left(r - \frac{1}{2}\sigma^2 m\right) \cdot T\right).$$

This proves the corollary.

Table 1. Gearing parameter for a standard call option and capped call options.

This table gives the value of the gearing parameter $\lambda_{K'}$ for a standard call option ($K'=+\infty$) and capped call options (K' ranging from 20% to 100 % of the initial value of the risky asset). The initial fund value is assumed to be equal to the initial value of the risky asset ($V_0=S_0$). The level of the fund guarantee is equal to the initial fund value ($K=V_0$). The maturity T of all call options is equal to 2 years. The annual risk-free interest rate r is equal to 5%. The annual volatility of the risky asset σ is equal to 20%.

K'	20%	40%	60%	80%	100%	$+\infty$
$\lambda_{K'}$	105.40%	93.04%	90.45%	89.67%	89.41%	89.28%

Table 2. Descriptive statistics of fund returns.

This table give descriptive statistics about gross returns of the risky asset used in the fund (Panel A), of the funds managed with option method (Panel B) and with the cushion method (Panel C). To compute the initial option price, the risk-neutral process of the risky asset price is assumed to be a Brownian motion with an annual risk-free rate of 5% and an annual volatility of 20 %. An annual risk premium of 5% is taken to simulate the historical distribution and then compute statistics.

Panel A. Risky asset

Mean	Standard deviation	Kurtosis	Top 5% quantile	Top 1% quantile
121.82%	34.05%	1.48	186%	234%

Panel B. Option method

	Call option cap *K'*					
	20%	40%	60%	80%	100%	$+\infty$
Mean	113.41%	115.02%	115.56%	115.83%	116.02%	116.14%
Standard deviation	8.49%	15.89%	19.43%	21.35%	22.55%	23.45%
Kurtosis	-1.30	-1.35	-0.13	1.28	2.73	4.76
Top 5% quantile	120%	140%	160%	166%	166%	166%
Top 1% quantile	120%	140%	160%	180%	200%	208%

Panel C. Cushion method

	Investment constraint *b*					
	20%	40%	60%	80%	100%	$+\infty$
Mean	112.71%	114.06%	114.75%	115.11%	115.41%	116.43%
Standard deviation	6.10%	11.88%	16.15%	19.46%	22.20%	35.44%
Kurtosis	-0.00	0.50	2.47	5.22	8.39	66.55
Top 5% quantile	124%	138%	148%	158%	162%	162%
Top 1% quantile	130%	152%	172%	194%	212%	298%

Figure 1. Final fund value with the option method.

This figure gives the final fund value as a function of the risky asset price at fund maturity in the case of a fund managed with the option method. We consider a standard (uncapped) call option ($K'=+\infty$) and capped call options (K' ranging from 20% to 100 % of the initial value of the risky asset). The maturity T of all call options is equal to the fund maturity (2 years). To compute the initial option price, the risk-neutral process of the risky asset price is assumed to be a Brownian motion with an annual risk-free rate of 5% and an annual volatility of 20 %. An annual risk premium of 5% is taken to simulate the historical distribution.

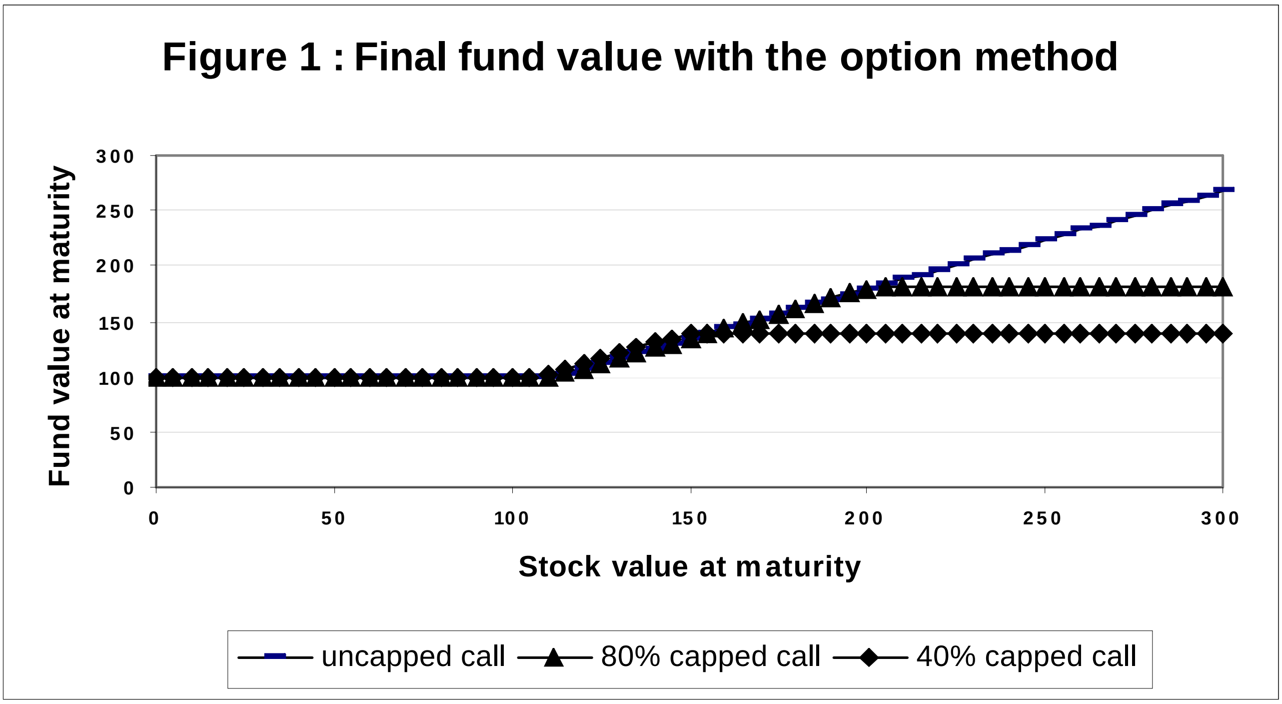


Figure 2. Final fund value with the cushion method.

This figure gives the final fund value at fund maturity in the case of a fund managed with the cushion method. For a given stock value at maturity, for the constrained strategies, the mean fund value is represented as it is path-dependent. We consider a standard (unconstrained) dynamic trading strategy (b=+∞) and constrained strategies (the maximum invested in the risky asset, b, ranging from 20% to 100% of the fund value). The fund maturity is equal to 2 years. The historical process of the risky asset price is assumed to be a Brownian motion with an annual expected return of 10% and an annual volatility of 20 %.

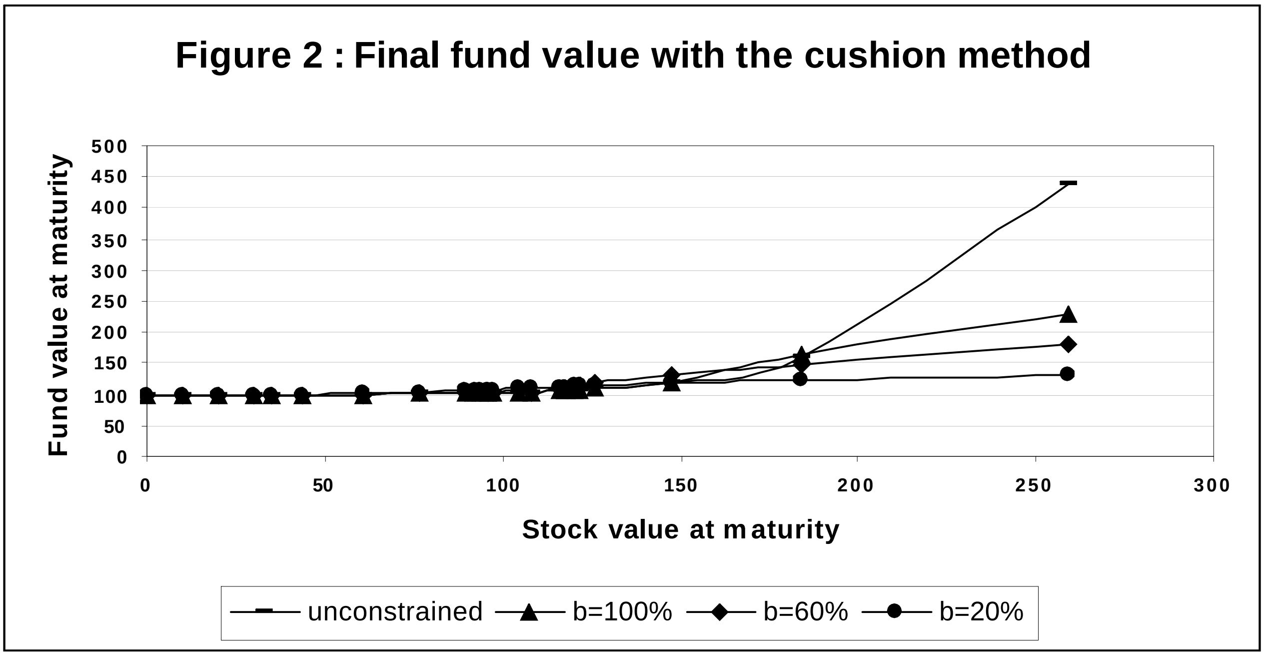


Figure 3. Fund behaviour in a bullish market.

This figure gives the fund value over time in the case of a bullish market in the case of a fund managed with the option method (Figure 3A) and in the case of a fund managed with the cushion method (Figure 3B). The risky asset is used as the underlying asset for the option and for the risky investment of the fund managed with cushion method. To compute the option price during the life of the fund, the risk-neutral process of the risky asset price is assumed to be a Brownian motion with an annual risk-free rate of 5% and an annual volatility of 20 %. An annual risk premium of 5% is taken to simulate the historical evolution of the risky asset price. The solid line represents the evolution of the risky asset value, which is associated with an unprotected buy-and-hold strategy. For funds managed with the option method, the current fund values are deduced by applying the Black-Scholes model as described in Appendix 1. For funds managed with the cushion method, the current values are obtained using the Milstein discretisation scheme of Equation (10).

Figure 3A : Evolution of the fund value with the option method in a bullish market

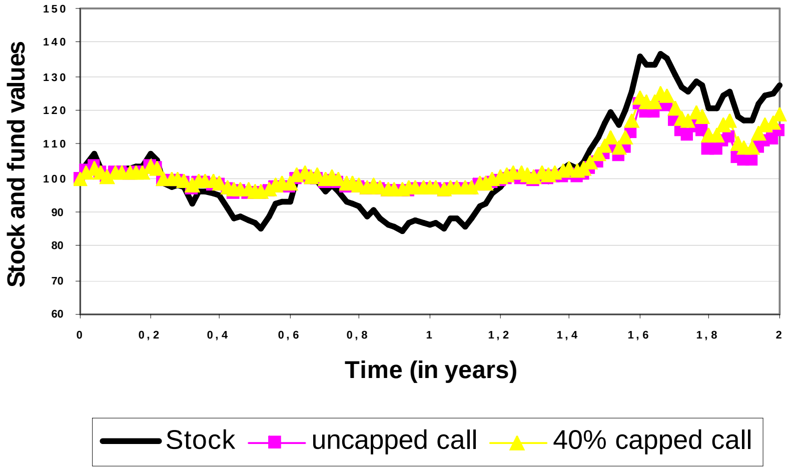


Figure 3B : Evolution of the fund value with the cushion method in a bullish market

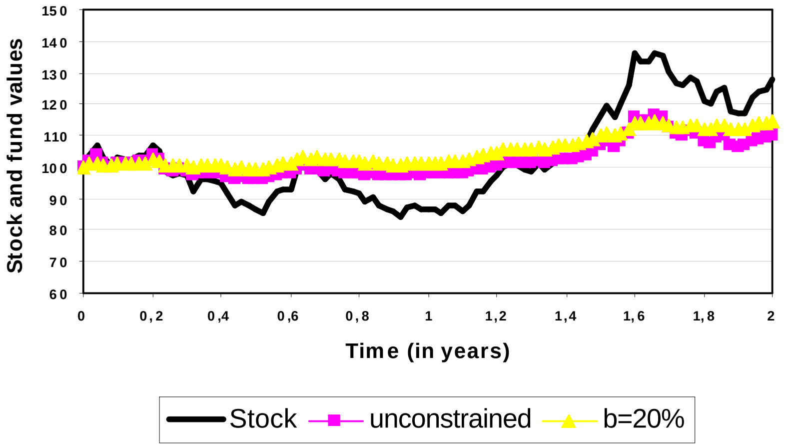
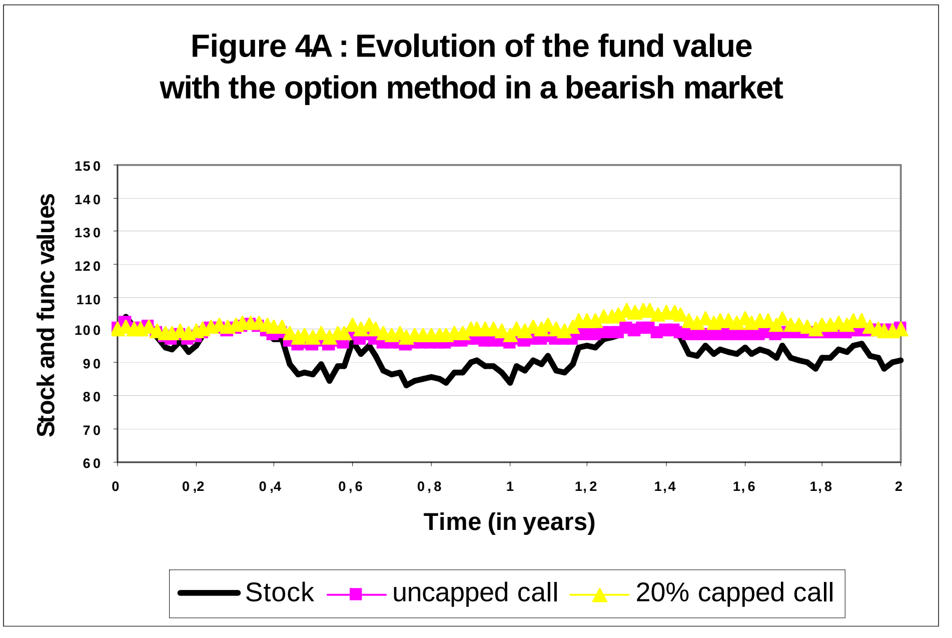


Figure 4. Fund behaviour in a bearish market.

This figure gives the fund value over time in the case of a bearish market in the case of a fund managed with the option method (Figure 4A) and in the case of a fund managed with the cushion method (Figure 4B). The risky asset is used as the underlying asset for the option and for the risky investment of the fund managed with cushion method. To compute the option price during the life of the fund, the risk-neutral process of the risky asset price is assumed to be a Brownian motion with an annual risk-free rate of 5% and an annual volatility of 20 %. An annual risk premium of 5% is taken to simulate the historical evolution of the risky asset price. The solid line represents the evolution of the risky asset value, which is associated with an unprotected buy-and-hold strategy. For funds managed with the option method, the current fund values are deduced by applying the Black-Scholes model as described in Appendix 1. For funds managed with the cushion method, the current values are obtained using the Milstein discretisation scheme of Equation (10).



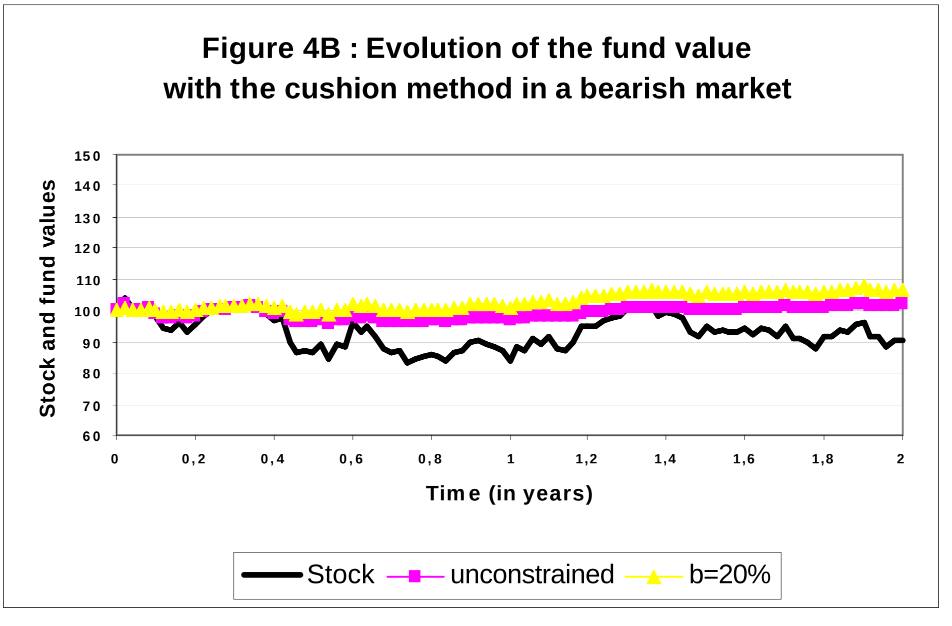
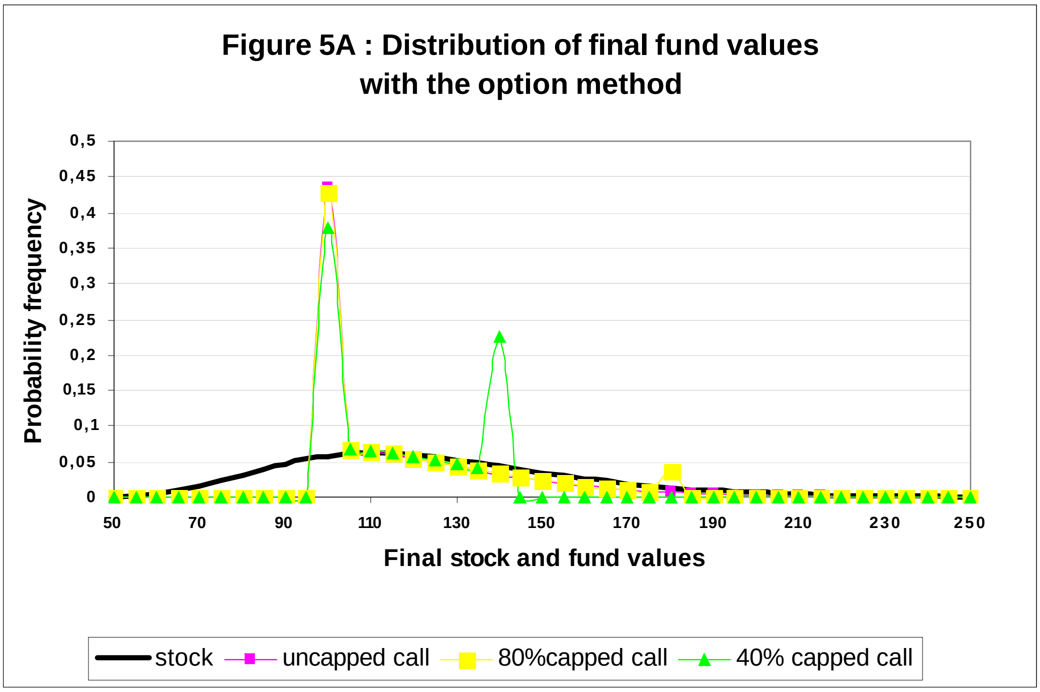


Figure 5. Distribution of the final fund value.

This figure gives the distribution of the fund value at maturity in the case of a fund managed with the option method (Figure 5A) and in the case of a fund managed with the cushion method (Figure 5B). Each distribution is obtained from 4.000 simulations of the risky asset price. The process of the risky asset price is assumed to be a Brownian motion. The log-normal distribution of the risky asset price at fund maturity is plotted for comparison.



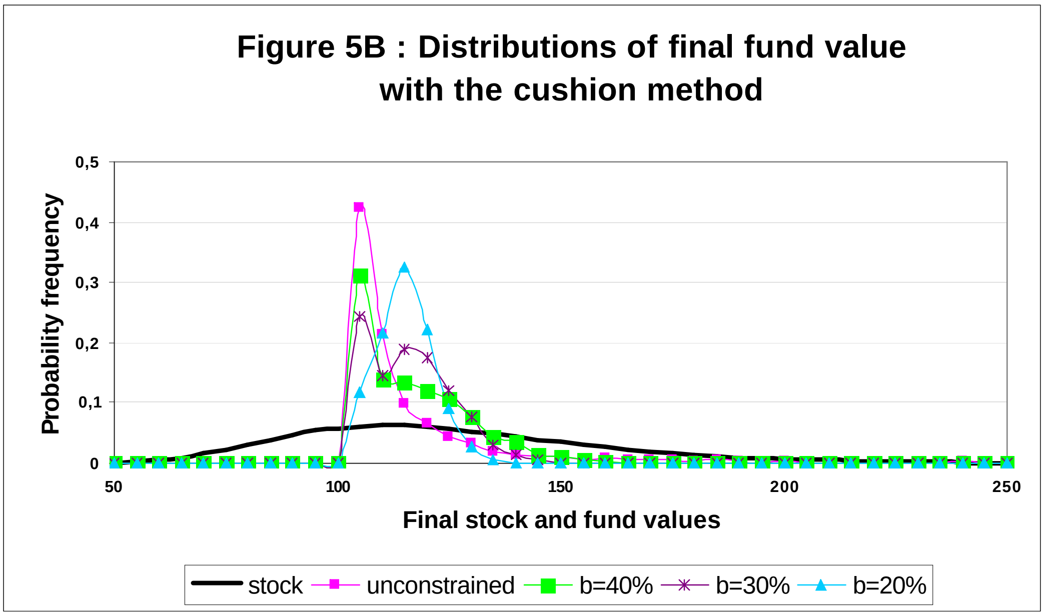


Figure 6. Fund performance and risk measured by the mean and standard deviation.

This figure plots the mean and standard deviation of funds managed with the option method and funds managed with the cushion method. For the option method we consider a standard (uncapped) call option ($K'=+\infty$) and capped call options (the cap value K' ranging from 20% to 100 % of the initial risky asset price). For the cushion method we consider a standard (unconstrained) dynamic strategy ($b=+\infty$) and constrained strategies (the maximum invested in the risky asset b ranging from 20% to 100% of the fund value). The maturity T of all funds is equal to 2 years. The mean and standard deviation of the funds are computed from statistical distributions obtained with the following parameters (annual values): 5% for the risk-free interest rate, 5% for the risk premium of the risky asset, and 20% for the volatility of the risky asset.

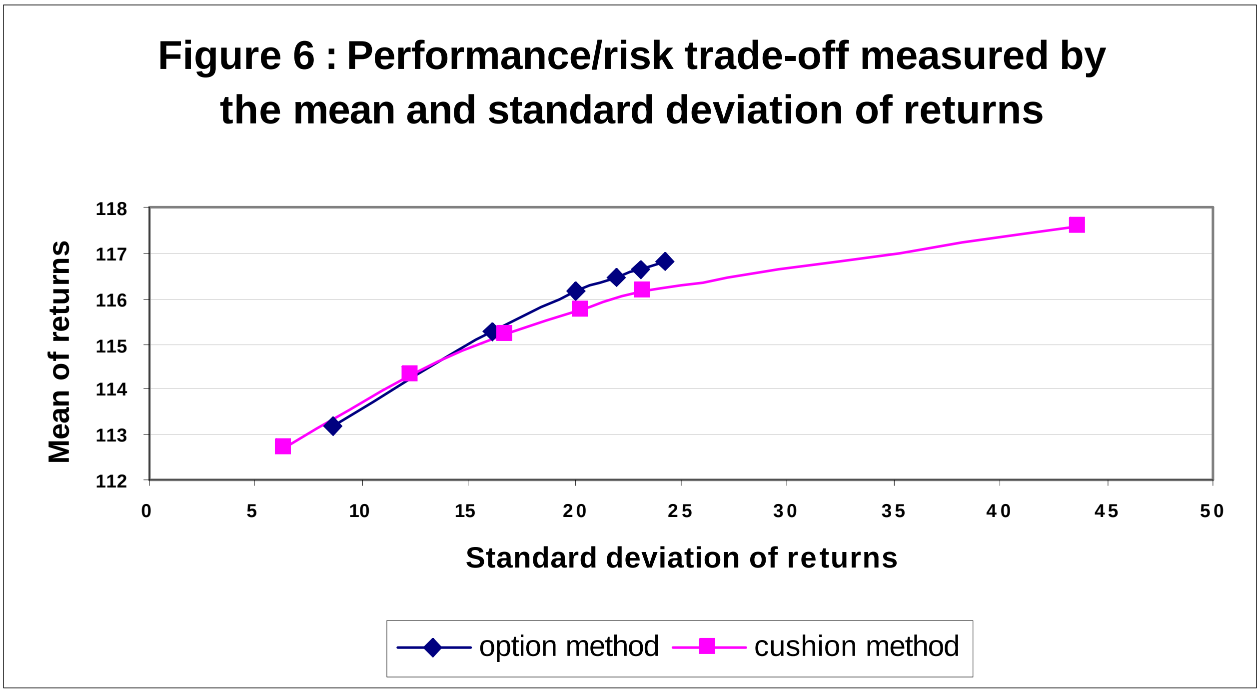


Figure 7. Fund performance and risk measured by the top quantile and standard deviation.

This figure plots the top quantile and standard deviation of funds managed with the option method and funds managed with the cushion method. For the option method we consider a standard (uncapped) call option ($K'=+\infty$) and capped call options (the cap value K' ranging from 20% to 100 % of the initial risky asset price). For the cushion method we consider a standard (unconstrained) dynamic strategy ($b=+\infty$) and constrained strategies (the maximum invested in the risky asset b ranging from 20% to 100% of the fund value). The maturity T of all funds is equal to 2 years. The top 5% and 1% quantiles and standard deviation of the funds are computed from statistical distributions obtained with the following parameters (annual values): 5% for the risk-free interest rate, 5% for the risk premium of the risky asset, and 20% for the volatility of the risky asset.

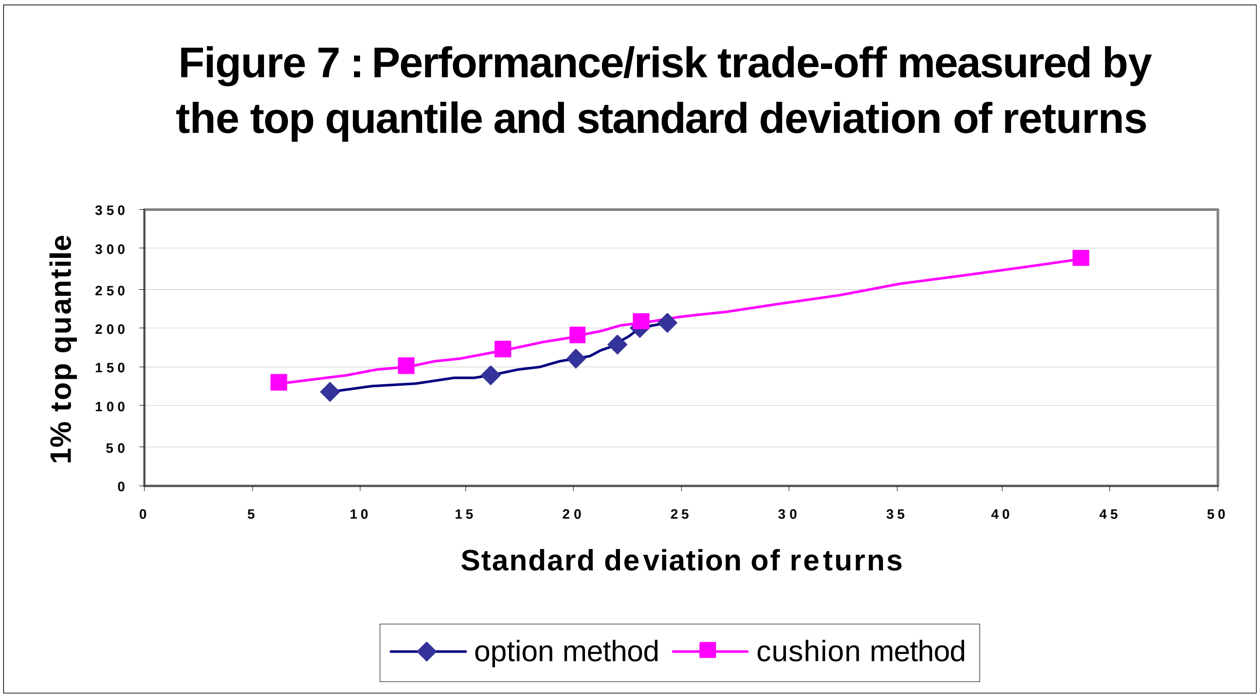


Figure 8. Fund performance and risk measured by the mean and absolute deviation.

This figure plots the mean and absolute deviation of funds managed with the option method and funds managed with the cushion method. For the option method we consider a standard (uncapped) call option (K'=+∞) and capped call options (the cap value K' ranging from 20% to 100 % of the initial risky asset price). For the cushion method we consider a standard (unconstrained) dynamic strategy (b=+∞) and constrained strategies (the maximum invested in the risky asset b ranging from 20% to 100% of the fund value). The maturity T of all funds is equal to 2 years. The mean and absolute deviation of the funds are computed from statistical distributions obtained with the following parameters (annual values): 5% for the risk-free interest rate, 5% for the risk premium of the risky asset, and 20% for the volatility of the risky asset.