On the Future of Statistics of Extremes:
Some Areas which Need Further Development

Miguel de Carvalho
Pontificia Universidad Católica de Chile
http://www.mat.puc.cl/∼mdecarvalho
MdeCarvalho@mat.puc.cl

Abstract

In this short essay, I resume some viewpoints that I shared with the Discussion Group 'Future of Statistics of Extremes' at the ESSEC *Conference on Extreme Events in Finance*, Royaumont Abbey, France, on December 15–17, 2014 (*extreme-events-in-finance.essec.edu*).[1]

Keywords. Asymptotic (in)dependence; Dimension reduction; Dimension reduction for dependence structure; Multivariate extremes; Multivariate spatial extremes; Nonstationary dependence structures; Prior-elicitation.

1 Nonstationary Dependence Structures

In many settings of applied interest, it seems natural to regard risk from a conditional viewpoint, and this leads us to ideas of 'conditional risk.' The comic in Figure 1 gives an interesting portrait on this matter.

In some cases we want to assess the risk of observing the occurrence of simultaneously large values of two variables (say two simultaneous large losses in a portfolio), and the mathematical basis for such modeling is that of statistics of multivariate extremes. In this context, 'extremal dependence' is often interpreted as a synonym of risk. However, if we want to develop ideas of 'conditional risk' for multivariate extremes, that is, if we want to assess systematic variation of risk in terms of covariates, we need to allow for nonstationary extremal dependence structures.

Modeling nonstationarity in marginal distributions has been the focus of much recent literature in applied extreme value modelling; see for instance Coles (2001, Ch. 6). The simplest approach in this setting was popularized long ago by Davison and Smith (1990), and it is based on indexing the location and scale parameters of the generalized extreme value distribution by a predictor, $x \in \mathcal{X}$,

$$G(y; \mu_x, \sigma_x, \xi) = \exp[-\{1 + \xi(y - \mu_x)/\sigma_x\}^{-1/\xi}]. \tag{1}$$

[1]I would like to thank, without implicating, Holger Rootzén and Ross Leadbetter for helpful comments on the penultimate version of this document, and to François Longin for encouraging discussion group participants to write down their viewpoints. I would like to thank other conference participants including Isabel Fraga Alves, Jan Beirlant, Frederico Caeiro, Ivette Gomes, Serguei Novak, Michał Warchoł, Chen Zhou, among others, for stimulating discussions and for pointing out many fascinating directions for the future of statistics of extremes. This work was funded by the Fondecyt project 11121186.

Figure 1: A comic on 'conditional risk' downloaded from *http://xkcd.com/795/*

When we consider bivariate extremes, it is well-known that the bivariate extreme value distribution, depends on an infinite-dimensional parameter (H) (Coles, 2001, Theorem 8.1), and it can be written as

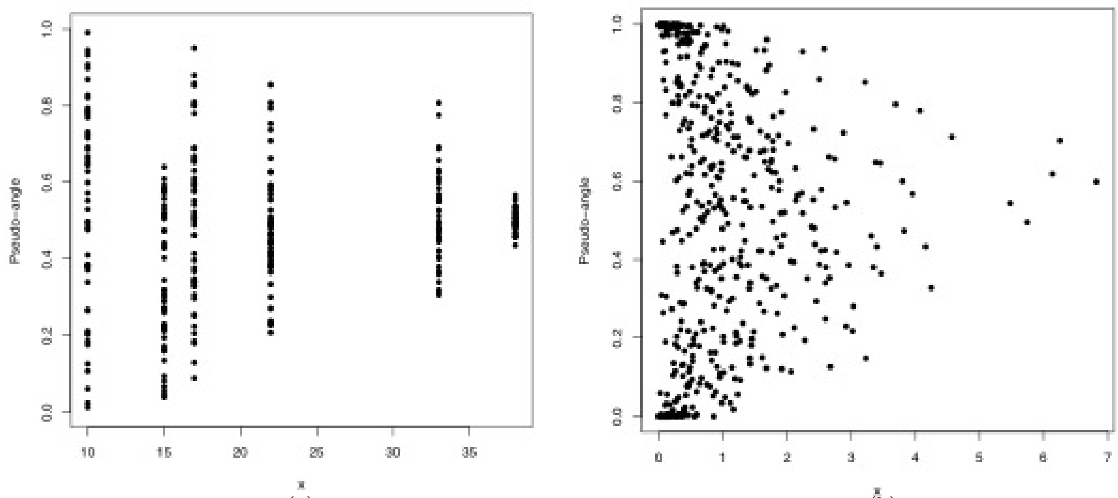
$$G(y_1, y_2; H) = \exp\left\{ -2 \int_0^1 \max\left(\frac{w}{y_1}, \frac{1-w}{y_2}\right) \mathrm{d}H(w)\right\}, \quad y_1, y_2 > 0,$$

where H is the so-called spectral distribution function, which is a distribution function on $[0, 1]$ obeying the moment constraint $\int_0^1 w \, \mathrm{d}H(w) = 1/2$. If H is differentiable, we define the spectral density as $h(w) = \mathrm{d}H/\mathrm{d}w$.

And how to model 'nonstationary bivariate extremes' if one must? Surprisingly, by comparison to the marginal case, approaches to modelling nonstationarity in the extremal dependence structure have received relatively little attention. These should be important to assess the dynamics governing extremal dependence of variables of interest. For example: Do we believe extremal dependence between returns of CAC and DAX has been constant over time, or can this level be changing over the years? To my knowledge there is a shortage of models for addressing this problem on the multivariate and spatial settings. Huser and Genton (2015) are currently developing research on this direction for the spatial setting. In terms of multivariate extremes, de Carvalho and Davison (2014) proposed a model for a family of spectral distribution functions $\{H_k : k = 0, \dots, K\}$, where each member of the family is linked to a predictor value. Their approach allows for modeling extremal dependence in settings such as in Figure 1 (a) but it excludes settings such as in Figure 1 (b).

Castro and de Carvalho (2015) and Castro, de Carvalho, and Wadsworth (2015) are currently developing models for these contexts, but there are still plenty of opportunities here. Their approaches are based on indexing the parameter of the bivariate extreme value distribution (H) with a covariate, i.e. considering $\{H_x : x \in \mathcal{X}\}$

$$G(y_1, y_2; H_x) = \exp\left\{ -2 \int_0^1 \max\left(\frac{w}{y_1}, \frac{1-w}{y_2}\right) \mathrm{d}H_x(w)\right\},$$



(a) (b)

Figure 2: Scatterplots presenting a configuration of data (predictor, pseudo-angles) for which the spectral density ratio model of Carvalho and Davison (2014) has been developed (a), and for a setting where it no longer applies (b).

which can be regarded as the analogue to bivariate setting of the approach in (1); the same comment applies to de Carvalho and Davison (2014).

2 "In Praise of Simplicity not Mathematistry!"

Theory and methods are the backbone of our field, without regular variation we wouldn't have gone far anyway. But, beyond theory, should our community be investing even more than it already is, in modeling and applications? As put simply by Box (1979), "all models are wrong, but some are useful." However, while most of us agree that models only provide an approximation to reality, we seem to be very demanding about the way that we develop theory about such (wrong yet useful) models. Some models entail ingenious approximations to reality, and yet are very successful in practice. For example, Cox (1972) is the most cited paper in statistics, and an important assumption in this model is that of proportional hazards. *Should we venture more on this direction in the future?* Applied work can also motivate new, and useful, theory. *Should we venture more on collaborating with researchers from other fields, or on creating more conferences such as this one, where one has the opportunity to regard risk and extremes from a broader perspective, so to think out of the box?*

3 Asymptotic (In)Dependence

Here, I comment on the need for further developing models compatible with both asymptotic dependence and asymptotic independence.

In two influential papers, Poon et al (2003, 2004) put forward that asymptotic independence was often observed on pairs of financial losses. This had important consequences in finance, mostly because inferences in a seminal paper (Longin and Solnik, 2001) had been based on the assumption of asymptotic dependence, and hence perhaps risk had been overestimated earlier. However, an important questions is:“What if pairs of financial losses can move over time from asymptotic independence to asymptotic dependence, and the other way around?”

Some markets are believed to be more integrated these days than in the past, so for such markets it is relevant to ask whether they could have entered an ’asymptotic dependence regime.’ An accurate answer to this question, would require however models able to allow for smooth transitions from asymptotic independence to asymptotic dependence, and vice versa, but as already mentioned in §1 at the moment there is a shortage of models for nonstationary extremal dependence structures.

Wadsworth et al (2015) presents an interesting approach for modeling asymptotic (in)dependence.

4 Spatial Multivariate Extremes

An important reference here is Genton et al. (2015), but there is a wealth of problems to work in this direction, so I stop my comment here.

5 Dimension Reduction

Is there a way to reduce dimension in such a way that the interesting features of the data—in terms of tails of multivariate distributions—are preserved?[2] I think it is fair to say that, apart from some exceptions, most models for multivariate extremes have been applied on the bivariate setting. I remember that at a seminal workshop on high-dimensional extremes, organized by Anthony C. Davison, at the Ecole Polytechnique Fédérale da Lausanne (14–18 September, 2009), for most talks high-dimensional actually meant ‘two-dimensional,’ and participants were all top scientists in the field.

Principal Component Analysis (PCA) itself would seem inappropriate, since principal axes are constructed in a way to find the directions that account for most variation, and for our axes of interest (whatever they are...) variation does not seem to be the most reasonable objective? A sensible approach could be however to use PCA for compositional data (Jolliffe, 2002, §13.3) and apply it to the pseudo-angles themselves; more on this below. More work should be perhaps be devoted to dimension reduction methods for multivariate extremes? There seem to be at least two main directions of interest:

1. **Dimension reduction for the dependence structure**: From this viewpoint, a sensible approach could be on adjusting existing methods of PCA (Principal Component Analysis) for compositional data (Jolliffe, 2002, §13.3) and apply them to the pseudo-angles, $W_i \in S_D$, for $i = 1, \hat{a}, n$, where S_D denotes the unit simplex in $\mathbb{R}^D$, i.e., $S_D = \{w \in \mathbb{R}^D : \sum_{d=1}^D w_d = 1,\ w_d \geq 0,\ d = 1, \ldots, D\}$. Such approach could provide a simple way to disentangle dependence into components that could practical interest.

[2] An interesting paper on dimension reduction for multivariate extremes appeared in the meantime at the *Electronic Journal of Statistics* (Chautru, 2015), after the discussion took place.

2. **Dimension reduction for number of variables directly**: Discussions with David Kraus (University Hospital Lausanne, CHUV, Switzerland) suggest that the concept of extremal stochastic integral (Stoev and Taqqu, 2005) could be important for developing analogues of the Karhunen–Loève expansion, but generating a decomposition in terms of maxima rather with a sum.

6 Prior Elicitation in Contexts where a Conflict of Interest Exists

How can we accurately elicit prior information when modeling extreme events in finance, given that in some cases a conflict of interest may exist? Suppose the case where a regulator demands an estimate to a Bank. If the Bank was better off by misreporting, collecting prior information from a Bank expert, this would obviously compromise the accuracy of the inference. In such cases, I think the only Bayesian analysis a regulator should be willing to accept should be an objective Bayes; see Berger (2006) for a review on objective Bayes.

REFERENCES

Berger, J. (2006), "The Case For Objective Bayesian Analysis," *Bayesian Analysis*, 1, 385–402.

Box, G. E. P. (1979), "Some Problems of Statistics and Everyday Life," *Journal of the American Statistical Association*, 74, 1–4.

Castro, D., de Carvalho, M., and Wadsworth, J. (2015), "Predictor-Dependent Modeling for Bivariate Extremes: Modeling and Estimation," Preprint.

Castro, D., and de Carvalho, M. (2015), "Spectral Density Regression for Bivariate Extremes," Preprint.

Chautru, E. (2015), "Dimension Reduction in Multivariate Extreme Value Analysis," *Electronic Journal of Statistics*, 9, 383–418.

Coles, S. (2001), *An Introduction to Statistical Modeling of Extreme Values*, New York: Springer.

Cox, D. R. (1972), "Regression Models and Life-Tables," *Journal of the Royal Statistical Society, Ser.* B, 34, 187–220.

Davison, A. C. and Smith, R. L. (1990), "Models for Exceedances over High Thresholds," *Journal of the Royal Statistical Society, Ser.* B, 52, 393–442.

de Carvalho, M., and Davison, A. C. (2014), "Spectral Density Ratio Models for Multivariate Extremes," *Journal of the American Statistical Association*, 109, 764–776.

Genton, M. G., Padoan, S. A., and Sang, H. (2015), "Multivariate Max-Stable Spatial Processes," *Biometrika*, 102, 215–230.

Huser, R., and Genton, M. G. (2014), "Non-Stationary Dependence Structures for Spatial Extremes," arXiv:1411.3174v1.

Jolliffe, I.T. (2002), *Principal Component Analysis*, New York: Springer.

Little, R. J. (2013), "In Praise of Simplicity not Mathematistry! Ten Simple Powerful Ideas for the Statistical Scientist," *Journal of the American Statistical Association*, 108, 359–369.

Longin, F., and Solnik, B. (2001), "Extreme Correlation of International Equity Markets," *The Journal of Finance*, 56, 649–676

Poon, S-H., Rockinger, M., and Tawn, J. (2003), "Modelling Extreme-Value Dependence in International Stock Markets," *Statistica Sinica*, 13, 929–953.

Poon, S-H., Rockinger, M., and Tawn, J. (2004), "Extreme Value Dependence in Financial Markets: Diagnostics, Models, and Financial Implications," *Review of Financial Studies*, 17, 2, 581–610.

Stoev, S., and Taqqu, M. S. (2005), "Extremal Stochastic Integrals: A Parallel Between Max-Stable Processes and Alpha-Stable Processes," *Extremes*, 8, 237–266.

Wadsworth, J. L., Tawn, J. A., Davison, A. C., and Elton, D. M. (2015), "Modelling Across Extremal Dependence Classes," Under revision.